Deformation Theory of Asymptotically Conical $Spin(7)$-Instantons

Tathagata Ghosh

School of Mathematics, University of Leeds,
Woodhouse Lane, Leeds, LS2 9JT, UK
Email: T.Ghosh@leeds.ac.uk

Abstract

In this talk we discuss the deformation theory of instantons on asymptotically conical $Spin(7)$-manifolds where the instanton is asymptotic to a fixed nearly G_2-instanton at infinity. By relating the deformation complex with Dirac operators and spinors, we apply spinorial methods to identify the space of infinitesimal deformations with the kernel of the twisted negative Dirac operator on the asymptotically conical $Spin(7)$-manifold.

Finally we apply this theory to describe deformations of Fairlie-Nuyts-Fubini-Nicolai (FNFN) $Spin(7)$-instantons on $\mathbb{R}^8$, where $\mathbb{R}^8$ is considered to be an asymptotically conical $Spin(7)$-manifold asymptotic to the cone over S^7. We also calculate the virtual dimension of the moduli space using Atiyah-Patodi-Singer index theorem and the spectrum of the twisted Dirac operator.[1]

Contents

[1] These notes are prepared for a talk at "Workshop BRIDGES: Special geometries and gauge theories" at University of Pau, France on 22nd June 2023.

1 Preliminaries

1.1 Nearly G_2- and $Spin(7)$-Manifolds

Definition 1.1. Let Σ be a Riemannian 7-dimensional manifold. A 3-form $\phi \in \Omega^3(\Sigma)$ is called a *G_2-structure on* Σ if in local orthonormal frame $e^1, \ldots, e^7$, ϕ can be written as

$$\phi = e^{127} + e^{347} + e^{567} + e^{145} + e^{136} + e^{235} - e^{246} \tag{1.1}$$

where $e^{ijk} := e^i \wedge e^j \wedge e^k$. The group G_2 is the stabilizer group of ϕ restricted to each tangent space, and is a 14-dimensional simple, connected, simply connected Lie group.

Definition 1.2. Let Σ be a 7-dimensional Riemannian manifold and $\phi \in \Omega^3(\Sigma)$. Then ϕ is called a *nearly (parallel) G_2-structure* on Σ if it satisfies

$$d\phi = \tau_0 \psi \tag{1.2}$$

where $\psi = *\phi$ and $\tau_0 \in \mathbb{R} \setminus \{0\}$. In this case, (Σ, ϕ) is called a *nearly G_2-manifold.*

Definition 1.3. Let X be an 8-dimensional Riemannian manifold equipped with a 4-form $\Phi \in \Omega^4(X)$ such that in local orthonormal basis $e^0, e^1, \ldots, e^7$, we have $\Phi = e^0 \wedge \phi + \psi$ where ϕ is as in (1.1) and $*(e^0 \wedge \phi) = \psi$. Then Φ is said to be a *$Spin(7)$-structure* on X and (X, Φ) is said to be an *almost $Spin(7)$-manifold.*

If Φ is torsion-free, i.e., if $\nabla\Phi = 0$ where ∇ is the Levi–Civita connection, or equivalently, if $d\Phi = 0$, then (X, Φ) is called a *$Spin(7)$-manifold.*

The 21-dimensional Lie group $Spin(7)$ is the stabiliser group of Φ restricted to each tangent space.

1.2 Asymptotically Conical $Spin(7)$-Manifolds

Let (Σ, g_Σ) be a Riemannian 7-manifold with a nearly G_2-structure ϕ satisfying $d\phi = 4\psi$ where $\psi = *\phi$. A *$Spin(7)$-cone* on Σ is $C(\Sigma) := (0, \infty) \times \Sigma$ together with a $Spin(7)$-structure $(C(\Sigma), \Phi_C)$ defined by

$$\Phi_C := r^3 dr \wedge \phi + r^4 \psi \tag{1.3}$$

where $r \in (0, \infty)$ is the coordinate. Σ is called the *link* of the cone. The metric g_C compatible with Φ_C is given by

$$g_C = dr^2 + r^2 g_\Sigma \tag{1.4}$$

We note that condition $d\phi = 4\psi$ implies the torsion free condition $d\Phi_C = 0$, which implies that $(C(\Sigma), g_C, \Phi)$ is a $Spin(7)$-manifold.

A $Spin(7)$-cone is not complete. Hence, we consider complete $Spin(7)$-manifolds whose geometry is asymptotic to the given (incomplete) G_2-cone.

Definition 1.4. Let (X, g, Φ) be a non-compact *Spin*(7)-manifold. X is called an *asymptotically conical (AC) Spin*(7)*-manifold with rate* $\nu < 0$ if there exists a compact subset $K \subset X$, a compact connected nearly G_2 manifold Σ, and a constant $R > 1$ together with a diffeomorphism

$$h : (R, \infty) \times \Sigma \to X \setminus K \tag{1.5}$$

such that

$$\left| \nabla_C^j (h^*(\Phi|_{X \setminus K}) - \Phi_C) \right| (r, p) = O(r^{\nu - j}) \quad \text{as } r \to \infty \tag{1.6}$$

for each $p \in \Sigma$, $j \in \mathbb{Z}_{\geq 0}$, $r \in (R, \infty)$; where ∇_C is the Levi–Civita connection for the cone metric g_C on $C(\Sigma)$, and the norm is induced by the metric g_C.

$X \setminus K$ is called the *end* of X and Σ the *asymptotic link* of X.

1.3 Lockhart–McOwen Analysis on AC *Spin*(7)-manifold

Let $\pi : E \to X$ be a vector bundle over X with a fibre-wise metric and a connection ∇ compatible with the metric.

Definition 1.5. Let $p \geq 1, k \in \mathbb{Z}_{\geq 0}, \nu \in \mathbb{R}$ and $C_c^\infty(E)$ be the space of compactly supported smooth sections of E. We define the *conically damped* or *weighted Sobolev space* $W_\nu^{k,p}(E)$ of sections of E over X of weight ν as follows:
For $\xi \in C_c^\infty(E)$, we define the *weighted Sobolev norm* $\| \cdot \|_{W_\nu^{k,p}(E)}$ as

$$\|\xi\|_{W_\nu^{k,p}(E)} = \left(\sum_{j=0}^{k} \int_X \left| \varrho^{-\nu+j} \nabla^j \xi \right|^p \varrho^{-8} \, \mathrm{dvol} \right)^{1/p} \tag{1.7}$$

The map $\varrho : X \to \mathbb{R}$, called a *radius function*, is defined by

$$\varrho(x) := \begin{cases} 1 & \text{if } x \in \text{the compact subset } K \subset X \\ r & \text{if } x = h(r, p) \text{ for some } r \in (2R, \infty),\ p \in \Sigma \\ \widetilde{r} & \text{if } x \in h((R, 2R) \times \Sigma) \end{cases} \tag{1.8}$$

where $h : (R, \infty) \times \Sigma \to X \setminus K$ is the diffeomorphism, and $\widetilde{r}$ is a smooth interpolation between its definition at infinity and its definition on K, in a decreasing manner. Then the weighted Sobolev space $W_\nu^{k,p}(E)$ is the completion of $C_c^\infty(E)$ with respect to the norm $\| \cdot \|_{W_\nu^{k,p}(E)}$.

We consider $E := \Lambda^* T^* X \otimes \mathfrak{g}_P$, and use the notation $\Omega_\nu^{m,k}(\mathfrak{g}_P) := W_\nu^{2,k}(\Lambda^m T^* X \otimes \mathfrak{g}_P)$.

2 Asymptotically Conical *Spin*(7)-Instantons

2.1 Asymptotically Conical *Spin*(7)-Instantons and Moduli Space

Definition 2.1. Let X be an AC *Spin*(7)-manifold asymptotic to the cone $C(\Sigma)$. Let $P \to X$ be a principal G-bundle over X. P is called *asymptotically framed* if there exists a principal bundle $Q \to \Sigma$ such that

$$h^* P \cong \pi^* Q$$

where $\pi : C(\Sigma) \to \Sigma$ is the natural projection.

Such framing always exists. So we fix a framing Q.

Definition 2.2. Let X be an AC $Spin(7)$-manifold asymptotic to the cone $C(\Sigma)$. Let $P \to X$ be an asymptotically framed bundle. A connection A on P is called an *asymptotically conical connection* with rate ν if there exists a connection A_Σ on $Q \to \Sigma$ such that

$$\left|\nabla_C^j (h^*(A) - \pi^*(A_\Sigma))\right| = O(r^{\nu-1-j}) \quad \text{as } r \to \infty \tag{2.1}$$

for each $p \in \Sigma$, $j \in \mathbb{Z}_{\geq 0}$, $\nu < 0$. The norm is induced by the cone metric and the metric on $\mathfrak{g}$.

A is called *asymptotic* to A_Σ and $\nu_0 := \inf\{\nu : A \text{ is AC with rate } \nu\}$ is called the *fastest rate of convergence of* A.

Let $G \to GL(V)$ be a faithful representation of G, and consider the associated vector bundle $\mathrm{End}(V)$. We define the *weighted gauge group* by

$$\mathcal{G}_{k+1,\nu} := \{\varphi \in C^0(\mathrm{End}(V)) : \|I - \varphi\|_{k+1,\nu} < \infty, \varphi \in G\}$$

We also define $\mathcal{G}_\nu := \bigcap_{l=1}^{\infty} \mathcal{G}_{l,\nu}$.

A connection A on P is a *$Spin(7)$-instanton* if the curvature F_A satisfies $*(\Phi \wedge F_A) = -F_A$. The *moduli space of $Spin(7)$-instantons asymptotic to A_Σ with rate ν* is given by

$$\mathcal{M}(A_\Sigma, \nu) := \{Spin(7) \text{ instanton } A \text{ on } P \text{ satisfying (2.1) asymptotic to } A_\Sigma\}/\mathcal{G}_\nu$$

2.2 Deformations of Asymptotically Conical $Spin(7)$-Instantons

Let A be an asymptotically conical reference connection that also satisfies the $Spin(7)$-instanton equation. Then, we have $\pi_7(F_A) = 0$. Now, we can write any other connection in some open neighbourhood of A as $A' = A + \alpha$ for $\alpha \in \Omega^1(\mathfrak{g}_P)$. Then,

$$F_{A'} - F_A = d_A\alpha + \frac{1}{2}[\alpha, \alpha].$$

Hence the connection A' is a $Spin(7)$-instanton if and only if $\pi_7(F_{A+\alpha}) = 0$, i.e.,

$$\pi_7\left(d_A\alpha + \frac{1}{2}[\alpha, \alpha]\right) = 0.$$

We also have the gauge fixing condition $d_A^*\alpha = 0$. We consider the non-linear operator

$$\begin{aligned} \mathfrak{D}_A^{\mathrm{NL}} : \Gamma(\Lambda^1 \otimes \mathfrak{g}_P) &\to \Gamma((\Lambda^0 \oplus \Lambda^2_7) \otimes \mathfrak{g}_P) \\ \alpha &\mapsto \left(d_A^*\alpha, \pi_7\left(d_A\alpha + \frac{1}{2}[\alpha, \alpha]\right)\right) \end{aligned} \tag{2.2}$$

Hence, the local moduli space of $Spin(7)$-instanton can be expressed as the zero set of $\mathfrak{D}_A^{\mathrm{NL}}$, i.e., $\left(\mathfrak{D}_A^{\mathrm{NL}}\right)^{-1}(0)$.

Let Σ be a nearly G_2-manifold and $Q \to \Sigma$ is a principal bundle. Let X be an AC $Spin(7)$-manifold with link Σ, and let $P \to X$ be an asymptotically framed bundle. Let A_Σ be a connection on Q. Consider the Dirac operators

$$\not{D}_{A_\Sigma} : \Gamma(\not{S}(\Sigma) \otimes \mathfrak{g}_Q) \to \Gamma(\not{S}(\Sigma) \otimes \mathfrak{g}_Q)$$

$$\not{D}_A^- : \Gamma(\not{S}^-(X) \otimes \mathfrak{g}_P) \to \Gamma(\not{S}^+(X) \otimes \mathfrak{g}_P)$$

Theorem 2.1. *The Dirac operator*

$$\not{D}_A^- : W^{k+1,2}_{\nu-1}(\not{S}^-(X) \otimes \mathfrak{g}_P) \to W^{k,2}_{\nu-2}(\not{S}^+(X) \otimes \mathfrak{g}_P)$$

is Fredholm if ν is not a critical weight, i.e., $\nu + \frac{5}{2} \in \mathbb{R} \setminus \operatorname{Spec} \not{D}_{A_\Sigma}$. Moreover, for two non-critical weights ν, ν' with $\nu \leq \nu'$, the jump in the index is given by

$$\operatorname{Index}_{\nu'} \not{D}_A^- - \operatorname{Index}_{\nu} \not{D}_A^- = \sum_{\nu < \lambda < \nu'} \dim \ker \left(\not{D}_{A_\Sigma} - \lambda - \frac{5}{2} \right).$$

Definition 2.3. For $\nu < 0$ the *space of infinitesimal deformations* is defined to be

$$\mathcal{I}(A, \nu) := \left\{ \alpha \in \Omega^{1,k+1}_{\nu-1}(\mathfrak{g}_P) : \not{D}_A^- \alpha = 0 \right\}. \tag{2.3}$$

The *obstruction space* $\mathcal{O}(A, \nu)$ is a finite-dimensional subspace of $\Omega^{0,k}_{\nu-2}(\mathfrak{g}_P) \oplus \Omega^{2,k}_{\nu-2}(\mathfrak{g}_P)$ such that,

$$\Omega^{0,k}_{\nu-2}(\mathfrak{g}_P) \oplus \Omega^{2,k}_{\nu-2}(\mathfrak{g}_P) = \not{D}_A^- \left(\Omega^{1,k+1}_{\nu-1}(\mathfrak{g}_P) \right) \oplus \mathcal{O}(A, \nu). \tag{2.4}$$

We note that $\mathcal{I}(A, \nu)$ and $\mathcal{O}(A, \nu)$ are precisely the kernel and cokernel of the twisted Dirac operator corresponding to the rate ν. We have the main theorem:

Theorem 2.2. *Let A be an AC $Spin(7)$-instanton asymptotic to a nearly G_2 instanton A_Σ. Moreover, let $\nu \in (\mathbb{R} \setminus \mathscr{D}(\not{D}_A^-)) \cap (-6, 0)$. Then there exists an open neighbourhood $\mathcal{U}(A, \nu)$ of 0 in $\mathcal{I}(A, \nu)$, and a smooth map $\kappa : \mathcal{U}(A, \nu) \to \mathcal{O}(A, \nu)$, with $\kappa(0) = 0$, such that an open neighbourhood of $0 \in \kappa^{-1}(0)$ is homeomorphic to a neighbourhood of A in $\mathcal{M}(A_\Sigma, \nu)$. Hence, the virtual dimension of the moduli space is given by $\dim \mathcal{I}(A, \nu) - \dim \mathcal{O}(A, \nu)$. Moreover, $\mathcal{M}(A_\Sigma, \nu)$ is a smooth manifold if $\mathcal{O}(A, \nu) = \{0\}$.*

3 Deformations of The FNFN $Spin(7)$-Instanton

Let us consider the asymptotically conical $Spin(7)$-manifold $\mathbb{R}^8$ asymptotic to the nearly G_2 manifold $\Sigma = S^7$. We consider S^7 as a homogeneous nearly G_2 manifold $Spin(7)/G_2$. Then we have the canonical bundle $G_2 \to Spin(7) \to S^7$ (call this bundle P). Also consider the bundle $Spin(7) \to Spin(7) \times_{(G_2,\iota)} Spin(7) \to S^7$ (call this bundle Q) where $\iota : G_2 \hookrightarrow Spin(7)$ is the inclusion. This bundle is (bundle) isomorphic to the trivial bundle $Spin(7) \to Spin(7) \times S^7 \to S^7$.

Let A_{flat} be a $Spin(7)$-invariant flat connection given by $A_{\text{flat}} = A_\Sigma + a$. Let $(r, \sigma) \in (0, \infty) \times S^7$. Then the connection $A(r, \sigma) = A_\Sigma(\sigma) + f(r)a(\sigma)$ where $f(r) = \dfrac{1}{Cr^2 + 1}$ for $C > 0$ is a function on $\mathbb{R}^8$, is an instanton on $\mathbb{R}^8$, shall be called the *FNFN $Spin(7)$-instanton*. Clearly the FNFN $Spin(7)$-instanton A is asymptotic to the canonical connection A_Σ with fastest rate of convergence -2.

3.1 The Main Result

Theorem 3.1. *The virtual dimension of the moduli space $\mathcal{M}(A_\Sigma, \nu)$ of the FNFN $Spin(7)$-instantons with decay rate $\nu \in (-2, 0) \setminus \{-1\}$ is given by*

$$\text{virtual-dim}\,\mathcal{M}(A_\Sigma, \nu) = \begin{cases} 1 & \textit{if} \;\; \nu \in (-2, -1) \\ 9 & \textit{if} \;\; \nu \in (-1, 0). \end{cases}$$

The dimensional jump of the moduli space happens for the rate -1, which corresponds to the eigenvalue $3/2$.

From the fact that the eigenvalues of the twisted Dirac operator in the range $[1/2, 5/2]$ are $1/2$ and $3/2$, corresponding to the trivial and spin representations respectively, we should expect that the rate of dilation should be $1/2 - 5/2 = -2$ and that of translation should be $3/2 - 5/2 = -1$. This can be easily verified from the fact that the two deformations translation and dilation are given by $\iota_{\frac{\partial}{\partial x^i}} F_A$ and $\iota_{x^i \frac{\partial}{\partial x^i}} F_A$ respectively.

3.2 Twisted Dirac Operators on $Spin(7)/G_2$

We start with a reductive homogeneous bundle G/H. Hence $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. Let $\{I_A\}$ is an orthonormal basis of $\mathfrak{g}$. We want to figure out the spectrum of the twisted Dirac operator

$$\not{\mathfrak{D}}_{A_\Sigma} : \Gamma(\not{\$}_{\mathbb{C}}(\Sigma) \otimes (\mathfrak{g}_P)_{\mathbb{C}}) \to \Gamma(\not{\$}_{\mathbb{C}}(\Sigma) \otimes (\mathfrak{g}_P)_{\mathbb{C}})$$

Using Frobenius reciprocity, we decompose the spinor bundle as

$$L^2(\not{\$}_{\mathbb{C}}(\Sigma) \otimes (\mathfrak{g}_P)_{\mathbb{C}}) \cong L^2(G, \Delta \otimes V)^H \cong \bigoplus_{\gamma \in \widehat{G}} \mathrm{Hom}(V_\gamma, \Delta \otimes V)^H \otimes V_\gamma \tag{3.1}$$

where $V_{(a,b,c)}$ is an irreducible representation of $\mathfrak{spin}(7)$ and $V_{(a,b)}$ is an irreducible representation of $\mathfrak{g}_2$.

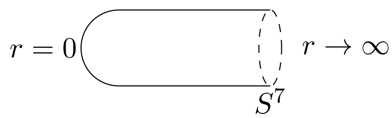
3.3 Eigenvalues of the Twisted Dirac Operator

For an FNFN $Spin(7)$-instanton, the fastest rate of convergence is -2. Hence we consider the family of moduli spaces $\mathcal{M}(A_\Sigma, \nu)$ for $\nu \in (-2, 0)$. We want to find the critical weights in $(-2, 0)$, i.e., $\nu \in (-2, 0)$ such that $\nu + \frac{5}{2} \in \mathrm{Spec}\, \not{\mathfrak{D}}_{A_\Sigma}$. Hence, we are interested in finding all the eigenvalues of the twisted Dirac operator in the interval $\left[-\frac{5}{2}, \frac{5}{2}\right]$. By an eigenvalue bound calculation, we find that if V_γ is not one of the irreducible representations $V_{(0,0,0)}$, $V_{(1,0,0)}$, $V_{(0,0,1)}$, $V_{(0,1,0)}, V_{(2,0,0)}$, $V_{(1,0,1)}$, then the operator $\left(\not{\mathfrak{D}}_{A_\Sigma}\right)_\gamma$ has no eigenvalues in the interval $\left(\frac{1}{2}, \frac{5}{2}\right)$.

Proposition 3.2. *By explicit computation, we find that the eigenvalues of the twisted Dirac operator $\left(\not{\mathfrak{D}}^0_{A_\Sigma}\right)_\gamma$ in the interval $\left[-\frac{5}{2}, \frac{5}{2}\right]$ are $-\frac{5}{2}, -\frac{3}{2}, \frac{1}{2}, \frac{3}{2}$, and that of in the interval $\left(\frac{1}{2}, \frac{5}{2}\right)$ is $\frac{3}{2}$ corresponding to the spin representation $V_{(0,0,1)}$.*

3.4 Index of the Twisted Dirac Operator

Let g_C be the asymptotically conical metric on $\mathbb{R}^8$. We define the metric $g_{CI} := \frac{1}{\rho^2} g_C$. Then $M := (\mathbb{R}^8, g_{CI})$ resembles a cigar (the reason g_{CI} is usually called a *cigar metric*).



Then,

$$\operatorname{Index}\left(\slashed{\mathfrak{D}}^-_{A,C} : W^{k,2}_{-\frac{7}{2}} \to W^{k-1,2}_{-\frac{9}{2}}\right) = \operatorname{Index}\left(\slashed{\mathfrak{D}}^-_{A,CI} : W^{k,2}_{CI} \to W^{k-1,2}_{CI}\right) \tag{3.2}$$

Proposition 3.3. *Let $B^8_R := \{x \in \mathbb{R}^8 : |x| \leq R\}$ be 8-dimensional ball of radius R. Then for sufficiently large R, we have*

$$\operatorname{Index}\left(\slashed{\mathfrak{D}}^-_{A,CI}, \mathbb{R}^8, g_{CI}\right) = \operatorname{Index}\left(\slashed{\mathfrak{D}}^-_{\widetilde{A},CI}, B^8_R, g_{CI}\right).$$

Moreover, for sufficiently large T, we have

$$\operatorname{Index}\left(\slashed{\mathfrak{D}}^-_{A}, \mathbb{R} \times S^7, g\right) = \operatorname{Index}\left(\slashed{\mathfrak{D}}^-_{\widetilde{A}'}, [-T, T] \times S^7, g\right).$$

where g is the cylindrical metric $g = dt^2 + g_{S^7}$.

Using Atiyah–Patodi–Singer index theorem us find that the index of the Dirac operator $\slashed{\mathfrak{D}}^-_{A_{CI}}$ on $\slashed{S}(\mathbb{R}^8, g_{CI})$ twisted by the bundle $\mathfrak{spin}(7)$ is

$$\begin{aligned}
\operatorname{Ind} \slashed{\mathfrak{D}}^-_{A_{CI}} &= I(M) + CS(S^7) + \frac{1}{2}\eta(S^7) \\
&= -\frac{1}{12}\int_{\mathbb{R}\times S^7} \left(p_1(\mathfrak{g}_P)^2 - 2p_2(\mathfrak{g}_P)\right) + 0 \\
&\quad + \frac{1}{12}\int_{\mathbb{R}\times S^7} \left(p_1(\mathfrak{g}_P)^2 - 2p_2(\mathfrak{g}_P)\right) \\
&= 0.
\end{aligned}$$

Hence,

$$\operatorname{Ind}_{-\frac{5}{2}} \slashed{\mathfrak{D}}^-_{A} = \operatorname{Ind} \slashed{\mathfrak{D}}^-_{A_{CI}} = 0.$$

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