Harmonic G_2-structures on almost Abelian Lie groups

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FAPESP - Fundação de Amparo à Pesquisa do Estado de São Paulo

Workshop BRIDGES: Specials geometries and gauge theories

June 21, 2023

Motivation

On a 7-dimensional Riemannian manifold M:

What is the best G_2-structure among some class?

- Among the all G_2-structures, it is the **torsion free** G_2-structure, since it corresponds with metrics with holonomy in G_2.
- Depending on the geometry/topology of M, the existence of torsion free a G_2-structure is trivial or obstructed.
- Sometimes is convenient to consider a weaker torsion condition. For instance, when M is a homogeneous space.

Outline

1. Review of G_2-structures
   G_2-structures and their torsion
   Harmonic G_2-structures
   Previous results

2. Almost Abelian Lie groups
   The torsion classes on $(\mathfrak{g}_A, \varphi)$
   The harmonicity of $(\mathfrak{g}_A, \varphi)$

G_2-structures and their torsion

A G_2-**structure** on a 7-manifold M is given by a differential 3-form φ on M, which is pointwise isomorphic to the 3-form

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356} \in \Lambda^3(\mathbb{R}^7)^*,$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$ and $(\mathbb{R}^7)^* = \langle \{e^1, \dots, e^7\} \rangle$.

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where $e^{ijk} = e^i \wedge e^j \wedge e^k$ and $(\mathbb{R}^7)^* = \langle\{e^1, \dots, e^7\}\rangle$.
For (M^7, φ), there are an induced Riemannian metric and orientation:

$$6 g_\varphi(u, v)\mathrm{vol}_\varphi = (u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi \quad \text{for} \quad u, v \in X(M).$$

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Hence, φ induces also:

- A Riemannian connection ∇_φ, a Hodge star operator $*_\varphi$, a dual 4-form $*_\varphi \varphi$.

And (M, φ) is called a G_2-**manifold** when:

$$\nabla_\varphi \varphi = 0 \quad (i.e. \quad \mathrm{Hol}(g_\varphi) \subseteq \mathrm{G}_2)$$

Fernández and Gray (1982)

φ is **torsion free** $\nabla_\varphi \varphi = 0$, if and only if, φ is **closed** $d\varphi = 0$ and **coclosed** $d * \varphi = 0$.

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The *intrinsic torsion* $\nabla\varphi$ is completely encoded by the *full torsion tensor* T:

$$\nabla_a \varphi_{bcd} = T_a^l (*\varphi)_{lbcd} \quad \text{where} \quad T \in \mathrm{End}(TM)$$

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According to the decomposition of $\mathcal{W} := \mathrm{End}(T_pM)$ into G_2-irreducible submodules ([Fernández-Gray, 1982])

$$\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4.$$

Where

$$\mathcal{W}_1 \oplus \mathcal{W}_3 \simeq \mathrm{sym}(T_pM) = [g_p] \oplus \mathrm{sym}_0(T_pM) \quad \text{and} \quad \mathcal{W}_2 \oplus \mathcal{W}_4 \simeq \mathfrak{so}(T_pM) = \mathfrak{g}_2 \oplus \mathbb{R}^7.$$

T splits into G_2-irreducible components [Karigiannis, 2008]:

$$T = \frac{\tau_0}{4} g - \frac{1}{2}\tau_2 - \frac{1}{4}\jmath(\tau_3) - *(\tau_1 \wedge *\varphi) \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4,$$

where $\jmath(\tau_3)_{ij} = *(e_i \lrcorner \varphi \wedge e_j \lrcorner \varphi \wedge \tau_3)$ and $\tau_k \in \Omega^k$ (for $k = 0, 1, 2, 3$) are called the *torsion forms*, and defined by:

$$d\varphi = \tau_0 * \varphi + 3\tau_1 \wedge \varphi + *\tau_3 \quad \text{and} \quad d * \varphi = 4\tau_1 \wedge *\varphi + \tau_2 \wedge \varphi.$$

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In total, there are 16-torsion classes, for instance:

- $\mathcal{W}_1$ is the class of nearly parallel G_2-structures.
- $\mathcal{W}_4$ is the class of locally conformal parallel G_2-structures.
- $\mathcal{W}_2$ is the class of closed G_2-structures.
- $\mathcal{W}_1 \oplus \mathcal{W}_3$ is the class of coclosed G_2-structures.

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Definition

A G_2-structure φ is called *harmonic* (divergence free) if $\mathrm{div}\, T = 0$.

Previous results

- The G_2-structure is harmonic if it has one of the following torsion [Grigorian, 2019]:
  (i) τ_0 is constant, $\tau_1 = 0$ and arbitrary τ_2 and τ_3.
  (ii) $\tau_0 = \tau_2 = \tau_3 = 0$ and τ_1 arbitrary.

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- Examples of harmonic G_2-structures on $\mathbb{R}^3 \ltimes_{A,B,C} \mathbb{R}^4$ with $A, B, C \in \mathfrak{sl}_4(\mathbb{R})$ such that $\tau_0 \neq 0$ and $\tau_1 \neq 0$ [Garrone 2021].
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- Spinorial description of the harmonic condition [Niedzialomski, 2020].
- General results on the associated gradient flow, from different perspectives:
  - By unit octonion sections [Grigorian, 2019].
  - Sections of a homogeneous fiber bundle [Loubeau- Sá Earp, 2019].
  - Evolving the G_2-structure φ [Dwivedi-Gianniotis-Karigiannis, 2019]. And recently, evolving the tensor field associated with the H-structure [Fadel-Loubeau-M.-Sá Earp, 2022].

Almost Abelian Lie algebras

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$$[e_7, e_j] = A(e_j) \quad \text{and} \quad [e_i, e_j] = 0 \quad \text{for} \quad e_i, e_j \in \mathfrak{h} \quad \text{and} \quad A \in \mathfrak{gl}(\mathfrak{h}).$$

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Consider the G_2-structure and the corresponding dual 4-form

$$\varphi = \omega \wedge e^7 + \rho^+ \quad \text{and} \quad *\varphi = \frac{\omega^2}{2} + \rho^- \wedge e^7,$$

where ω, ρ_+ and $\rho^- = *_{\mathfrak{h}}\rho^+$ are a $\mathrm{SU}(3)$-structure on $\mathfrak{h} \simeq \mathbb{R}^6$.

G_2-structures φ have been studied in Almost Abelian Lie algebras $\mathfrak{g}_A$:

1. $(\mathfrak{g}_A, \varphi)$ is closed, if and only if $A \in \mathfrak{sl}(\mathbb{C}^3)$ [Freibert 2012].
2. $(\mathfrak{g}_A, \varphi)$ is coclosed, if and only if $A \in \mathfrak{sp}(\mathbb{R}^6)$ [Freibert 2013].

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Consider the splitting

$$\mathfrak{gl}(\mathbb{R}^6) = \mathbb{R} \cdot I_6 \oplus \mathrm{sym}^0_+(\mathbb{R}^6) \oplus \mathrm{sym}^0_-(\mathbb{R}^6) \oplus \mathbb{R} \cdot J \oplus \mathfrak{su}(3) \oplus \mathfrak{m},$$

where

$$\begin{aligned}
\mathrm{sym}^0_+(\mathbb{R}^6) =& \{A \in \mathfrak{gl}(\mathbb{R}^6); \quad A^t = A, \quad \mathrm{tr}(A) = 0 \quad \text{and} \quad JA = AJ\} \\
\mathrm{sym}^0_-(\mathbb{R}^6) =& \{A \in \mathfrak{gl}(\mathbb{R}^6); \quad A^t = A \quad \text{and} \quad JA = -AJ\} \\
\mathfrak{su}(3) =& \{A \in \mathfrak{gl}(\mathbb{R}^6); \quad A^t = -A, \quad \mathrm{tr}(JA) = 0 \quad \text{and} \quad JA = AJ\} \\
\mathfrak{m} =& \{A \in \mathfrak{gl}(\mathbb{R}^6); \quad A^t = -A, \quad \text{and} \quad JA = -AJ\}.
\end{aligned}$$

For $A = S(A) + C(A) \in \mathfrak{gl}(\mathbb{R}^6)$, we have

$$A = \frac{\mathrm{tr}(A)}{6} I_6 + S_+(A) + S_-(A) + \frac{\mathrm{tr}(JA)}{6} J + C_+(A) + C_-(A),$$

where

$$S_+(A) \in \mathrm{sym}^0_+(\mathbb{R}^6), \quad S_-(A) \in \mathrm{sym}^0_-(\mathbb{R}^6), \quad C_+(A) \in \mathfrak{su}(3) \quad \text{and} \quad C_-(A) \in \mathfrak{m},$$

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For any k-form $\gamma \in \Lambda^k(\mathbb{R}^6)^*$, the Lie algebra $\mathfrak{gl}(\mathbb{R}^6)$ acts by

$$\theta(A)\gamma = \frac{d}{dt}|_{t=0}(e^{-At})^*\gamma.$$

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In particular, for $\omega \in \Lambda^2$ it satisfies

$$\theta(A)\omega = \frac{\mathrm{tr}(A)}{2}\omega + \theta(S_+(A))\omega + \theta(C_-(A))\omega \in \Lambda^2_1 \oplus \Lambda^2_8 \oplus \Lambda^2_6$$

The torsion classes on $(\mathfrak{g}_A, \varphi)$

Lemma

The 1-form $\alpha = - *_{\mathfrak{h}} (\theta(A^t)\omega \wedge \rho^-)$ on $\mathbb{R}^6$ satisfies the identity $\alpha^\sharp \lrcorner \rho_+ = -4JC_-(A)$ and

$$\alpha = 0 \quad \Leftrightarrow \quad C_-(A) = 0.$$

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Lemma

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Using the expressions $d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *\tau_3$ and $d\psi = 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi$, we obtain:

Proposition (M. 2022)

The torsion forms of $(\mathfrak{g}_A, \varphi)$ are:

$$\tau_0 = \frac{2}{7}\operatorname{tr}(JA), \quad \frac{1}{4}\jmath(\tau_3) = \frac{1}{14}\operatorname{tr}(JA)I_6 - JS_-(A) + \frac{1}{4}J\alpha^\sharp \odot e^7 - \frac{3}{7}\operatorname{tr}(JA)e^7 \otimes e^7,$$
$$\tau_1 = \frac{1}{12}\alpha - \frac{1}{6}\operatorname{tr}(A)e^7 \quad \tau_2 = \frac{2}{3}JC_-(A) - 2JS_+(A) - \frac{1}{3}J\alpha^\sharp \wedge e^7.$$

Class	Vanishing torsion	Bracket relation
$\mathcal{W} = \{0\}$	$\tau_0 = 0, \tau_1 = 0, \tau_2 = 0, \tau_3 = 0$	$A \in \mathfrak{su}(3)$
$\mathcal{W}_4$	$\tau_0 = 0, \tau_2 = 0, \tau_3 = 0$	$A \in \mathbb{R} \cdot I_6 \oplus \mathfrak{su}(3)$
$\mathcal{W}_2$	$\tau_0 = 0, \tau_1 = 0, \tau_3 = 0$	$A \in \mathrm{sym}^0_+ \oplus \mathfrak{su}(3)$
$\mathcal{W}_3$	$\tau_0 = 0, \tau_1 = 0, \tau_2 = 0$	$A \in \mathrm{sym}^0_- \oplus \mathfrak{su}(3)$
$\mathcal{W}_1 \oplus \mathcal{W}_3$	$\tau_1 = 0, \tau_2 = 0$	$A \in \mathrm{sym}^0_- \oplus \mathbb{R} \cdot J \oplus \mathfrak{su}(3)$
$\mathcal{W}_2 \oplus \mathcal{W}_4$	$\tau_0 = 0, \tau_3 = 0$	$A \in \mathbb{R} \cdot I_6 \oplus \mathrm{sym}^0_+ \oplus \mathfrak{su}(3)$
$\mathcal{W}_3 \oplus \mathcal{W}_4$	$\tau_0 = 0, \tau_2 = 0$	$A \in \mathbb{R} \cdot I_6 \oplus \mathrm{sym}^0_- \oplus \mathfrak{su}(3)$
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$\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	$\tau_0 = 0$	$A \in \mathrm{sym}(\mathbb{R}^6) \oplus \mathfrak{m} \oplus \mathfrak{su}(3)$
$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	No vanishing condition	$A \in \mathfrak{gl}(6, \mathbb{R})$

Table: Torsion classes of $(\mathfrak{g}_A, \varphi)$ [M. 2022]

From the torsion formulae:

- The Ricci curvature is $\mathrm{Ric}_A = \frac{1}{2}[A, A^t] - \mathrm{tr}(A)S(A) - \mathrm{tr}(S(A)^2)e^7 \otimes e^7$ [Arroyo, 2013]. The Lie algebra $(\mathfrak{g}_A, \varphi)$ does not induce an Einstein metric if $e_7 \lrcorner \tau_1 = 0$.

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- Notice that $\tau_3 = 0$ implies $\tau_0 = 0$. There does not exist an $(\mathfrak{g}_A, \varphi)$ with torsion strictly in one of the following classes:
  (i) $\mathcal{W}_1$ (in this class $\mathrm{Scal}(g) = \frac{28}{9}\tau_0^2$, but $(\mathfrak{g}_A, g)$ is either flat or $\mathrm{Scal}(g) < 0$ [Milnor, 1976]).

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  (iv) $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$ (..?)

From the torsion formulae:

- The Ricci curvature is $\mathrm{Ric}_A = \frac{1}{2}[A, A^t] - \mathrm{tr}(A)S(A) - \mathrm{tr}(S(A)^2)e^7 \otimes e^7$ [Arroyo, 2013]. The Lie algebra $(\mathfrak{g}_A, \varphi)$ does not induce an Einstein metric if $e_7 \lrcorner \tau_1 = 0$.
- Notice that $\tau_3 = 0$ implies $\tau_0 = 0$. There does not exist an $(\mathfrak{g}_A, \varphi)$ with torsion strictly in one of the following classes:
  - (i) $\mathcal{W}_1$ (in this class $\mathrm{Scal}(g) = \frac{28}{9}\tau_0^2$, but $(\mathfrak{g}_A, g)$ is either flat or $\mathrm{Scal}(g) < 0$ [Milnor, 1976]).
  - (ii) $\mathcal{W}_1 \oplus \mathcal{W}_2$ (If M is connected this class reduces to either $\mathcal{W}_1$ or $\mathcal{W}_2$ [Martin Cabrera-Monar-Swann, 1996])
  - (iii) $\mathcal{W}_1 \oplus \mathcal{W}_4$ (If φ is invariant then $\mathcal{W}_1 \oplus \mathcal{W}_4$ reduces to either $\mathcal{W}_1$ or $\mathcal{W}_4$).
  - (iv) $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$ (..?)
- The closed case

$$\tau_0 = 0 \quad \tau_1 = 0 \quad \text{and} \quad \tau_3 = 0 \quad \Leftrightarrow \quad A = S_+(A) + C_+(A) \in \mathfrak{sl}(\mathbb{C}^3).$$

  The coclosed case

$$\tau_2 = 0 \quad \text{and} \quad \tau_1 = 0 \quad \Leftrightarrow \quad A = S_-(A) + \frac{\mathrm{tr}(JA)}{6}J + C_+(A) \in \mathfrak{sp}(\mathbb{R}^6).$$

Definition

The Lie algebra $\mathfrak{g}$ is called a *unimodular* Lie algebra if $\mathrm{tr}(\mathrm{ad}(u)) = 0$ for every $u \in \mathfrak{g}$. A *lattice* Γ of a Lie group G is a discrete subgroup $\Gamma \subset G$, such that the quotient $\Gamma \backslash G$ is compact.

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Class	Vanishing torsion	Bracket relation
$\mathcal{W} = \{0\}$	$\tau_0 = 0, \tau_1 = 0, \tau_2 = 0, \tau_3 = 0$	$A \in \mathfrak{su}(3)$
$\mathcal{W}_2$	$\tau_0 = 0, \tau_1 = 0, \tau_3 = 0$	$A \in \mathrm{sym}^0_+ \oplus \mathfrak{su}(3)$
$\mathcal{W}_3$	$\tau_0 = 0, \tau_1 = 0, \tau_2 = 0$	$A \in \mathrm{sym}^0_- \oplus \mathfrak{su}(3)$
$\mathcal{W}_1 \oplus \mathcal{W}_3$	$\tau_1 = 0, \tau_2 = 0$	$A \in \mathrm{sym}^0_- \oplus \mathbb{R} \cdot J \oplus \mathfrak{su}(3)$
$\mathcal{W}_2 \oplus \mathcal{W}_3$	$\tau_0 = 0, \tau_1 = 0$	$A \in \mathrm{sym}^0_+ \oplus \mathrm{sym}^0_- \oplus \mathfrak{su}(3)$
$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$	$\tau_1 = 0$	$A \in \mathrm{sym}^0_+ \oplus \mathrm{sym}^0_- \oplus \mathbb{R} \cdot J \oplus \mathfrak{su}(3)$
$\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	$\tau_0 = 0$	$A \in \mathrm{sym}^0_+ \oplus \mathrm{sym}^0_- \oplus \mathfrak{m} \oplus \mathfrak{su}(3)$
$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	No vanishing condition	$A \in \mathrm{sym}^0_+ \oplus \mathrm{sym}^0_- \oplus \mathbb{R} \cdot J \oplus \mathfrak{m} \oplus \mathfrak{su}(3)$

Table: Torsion classes of $(\mathfrak{g}_A, \varphi)$ unimodular

The harmonicity of $(\mathfrak{g}_A, \varphi)$

The full torsion tensor of $(\mathfrak{g}_A, \varphi)$ is

$$T = \frac{1}{2}\left(\begin{array}{c|c} [J, S(A)] + [J, C(A)] + (JA^t + AJ) & -J\alpha(A)^\sharp \\ \hline 0 & \mathrm{tr}(JA) \end{array}\right)$$

The Levi-Civita connection given by the left-invariant metric [Milnor, 1976] is:

$$\nabla_7 e_7 = 0, \quad \nabla_i e_7 = -S(A)(e_i), \quad \nabla_7 e_i = C(A)(e_i) \quad \text{and} \quad \nabla_i e_j = \langle S(A)(e_i), e_j \rangle e_7$$

where $i, j = 1, ..., 6$.

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Proposition [M. 2022]

The divergence of T is

$$\mathrm{div}\, T = -\frac{1}{2}\,\mathrm{tr}(A) J^* \alpha(A) + \frac{1}{2}\theta(C(A)) J^* \alpha(A) - \frac{1}{2}\,\mathrm{tr}(A)\,\mathrm{tr}(JA) e^7$$

Theorem [M. 2022]

The almost Abelian Lie algebra with G_2-structure $(\mathfrak{g}_A, \varphi)$ is harmonic, if and only if,

$$\operatorname{tr}(A)\operatorname{tr}(JA) = 0 \quad \text{and} \quad JC(A)J(\alpha^\sharp) = -\operatorname{tr}(A)\alpha^\sharp.$$

In particular, φ is harmonic if its torsion belongs to one of the following classes:

$$\{0\}, \quad \mathcal{W}_2, \quad \mathcal{W}_3, \quad \mathcal{W}_4,$$
$$\mathcal{W}_1 \oplus \mathcal{W}_3, \quad \mathcal{W}_2 \oplus \mathcal{W}_4, \quad \mathcal{W}_3 \oplus \mathcal{W}_4,$$
$$\mathcal{W}_2 \oplus \mathcal{W}_3, \quad \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3.$$

Further, if φ is of type $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ and $\operatorname{div} T = 0$, then φ is of type $\mathcal{W}_1 \oplus \mathcal{W}_3$ or $\mathcal{W}_3 \oplus \mathcal{W}_4$.

- The torsion classes $\{0\}$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_4$ and $\mathcal{W}_2 \oplus \mathcal{W}_3$ are generically harmonic. Since τ_0 is constant for Lie groups, then the torsion classes $\mathcal{W}_1 \oplus \mathcal{W}_3$ and $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ are also harmonic.

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- The almost Abelian Lie algebras $(\mathfrak{g}_A, \varphi)$ whose G_2-structure has torsion in the classes $\mathcal{W}_2 \oplus \mathcal{W}_4$, and $\mathcal{W}_3 \oplus \mathcal{W}_4$ are new examples of harmonic G_2-structures. However, these new examples $(\mathfrak{g}_A, \varphi)$ do not admit a lattice.

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- The almost Abelian Lie algebras $(\mathfrak{g}_A, \varphi)$ whose G_2-structure has torsion in the classes $\mathcal{W}_2 \oplus \mathcal{W}_4$, and $\mathcal{W}_3 \oplus \mathcal{W}_4$ are new examples of harmonic G_2-structures. However, these new examples $(\mathfrak{g}_A, \varphi)$ do not admit a lattice.
- (G_2-structure with torsion in $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$) For

$$Ae_1 = Ae_2 = 0, \quad Ae_3 = e_5, \quad Ae_4 = -e_6, \quad Ae_5 = -e_3, \quad Ae_6 = e_4,$$

we have

$$\tau_0 = 0, \quad \tau_1 = 4e^2, \quad \tau_2 = -\frac{1}{3}\left(e^{36} + e^{45} - 4e^{17}\right) \quad \text{and} \quad \jmath(\tau_3) = -4(e^1 \otimes e^7 + e^7 \otimes e^1).$$

And $\alpha^\sharp = 4e_2$, since $J\alpha^\sharp \in \ker A$ then $\operatorname{div} T = 0$.

Many thanks