STABILITY OF PULLBACKS OF SHEAVES ALONG TORIC FIBRATIONS

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Abstract. Locally trivial fibrations and blowups are examples of fibrations. Given a reflexive sheaf over a polarized toric variety, we study the stability of its reflexive pullback along a toric fibration. Secondly, we will focus on the case of blowups along irreducible subvarieties of dimension greater than or equal to one.

These are the notes of a talk given at "Workshop BRIDGES: Special geometries and gauge theories". This talk is based on the paper [NT22].

1. Slope stability and Hermitian Yang-Mills connections

The notion of stability was first introduced by Mumford [Mum63] in his study of moduli spaces of vector bundles on curves. It was generalized in higher dimension by Takemoto [Tak72]. A vector bundle, or more generally a torsion-free sheaf $\mathscr{E}$ on a complex projective variety X is said to be *slope stable* (resp. *semistable*) with respect to a polarization L, if for any proper coherent subsheaf $\mathscr{F}$ of $\mathscr{E}$ with $0 < \mathrm{rk}(\mathscr{F}) < \mathrm{rk}(\mathscr{E})$, one has $\mu_L(\mathscr{F}) < \mu_L(\mathscr{E})$ (resp. $\mu_L(\mathscr{F}) \leq \mu_L(\mathscr{E})$) where the *slope* $\mu_L(\mathscr{E})$ of $\mathscr{E}$ with respect to L is given by

$$\mu_L(\mathscr{E}) = \frac{c_1(\mathscr{E}) \cdot L^{n-1}}{\mathrm{rk}(\mathscr{E})}.$$

We say that $\mathscr{E}$ is *polystable* if it is a direct sum of stable subsheaves of the same slope. Finally, $\mathscr{E}$ is said to be *unstable* with respect to L, if there is $\mathscr{F} \subsetneq \mathscr{E}$ such that $\mu_L(\mathscr{F}) > \mu_L(\mathscr{E})$.

Remark 1.1. If X is a compact Kähler manifold of dimension n with a Kähler metric ω, then the ω-slope of a torsion free sheaf $\mathscr{E}$ on X is defined by:

$$\mu_\omega(\mathscr{E}) = \frac{\deg_\omega(\mathscr{E})}{\mathrm{rk}(\mathscr{E})} = \frac{1}{\mathrm{rk}(\mathscr{E})} \int_X c_1(\mathscr{E}) \wedge \omega^{n-1}.$$

Let X be a compact Kähler manifold of dimension n with a Kähler metric ω. We denote by Λ_ω the adjoint of the Lefschetz operator.

Definition 1.2. Let $\mathscr{E}$ be a holomorphic vector bundle over X. A hermitian metric h on $\mathscr{E}$ is *Hermite-Einstein* with respect to ω if the curvature F_h of the corresponding Chern connection satisfies

$$\Lambda_\omega(\imath F_h) = c \cdot \mathrm{Id}_{\mathscr{E}}$$

for some constant c. If h is some hermitian metric on the smooth complex vector bundle underlying $\mathscr{E}$, a hermitian connection A on $(\mathscr{E}, h)$ is said to be *hermitian Yang-Mills* (HYM) if

$$F_A^{0,2} = 0 \qquad \text{and} \qquad \Lambda_\omega(\imath F_A) = c \cdot \mathrm{Id}_{\mathscr{E}}.$$

By the Donaldson-Uhlenbeck-Yau theorem [UY86] (or Kobayashi-Hitchin correspondence), if L is a polarization on X, then the existence of HYM connections on $\mathscr{E}$ with respect to $\omega \in c_1(L)$ is related to the polystability of $\mathscr{E}$ with respect to L. It is then natural to understand how HYM connections, or polystable bundles behave with respect to natural maps between complex polarized manifold.

In the case of immersions, the Metha-Ramanathan theorem [MR84, Theorem 4.3] asserts that the restriction of a slope (semi)stable torsion-free sheaf to a generic complete intersection of high degree remains slope (semi)stable. In this document, we address to the problem of pulling-back (semi)stable reflexive sheaves along fibrations.

Remark 1.3. According to Hartshorne [Har80], reflexive sheaves can be seen as vector bundles with singularities and their study gives a better description of vector bundles.

2. Existence of HYM connections

In the paper [ST22], Sektnan and Tipler studied a similar problem. They considered the following problem. Let $\pi : (X, H) \longrightarrow (B, L)$ be a holomorphic submersion between polarized compact complex manifolds such that:

- for any $b \in B$, $X_b = \pi^{-1}(b)$ is smooth;
- L is an ample line bundle on B;
- H is a relatively ample line bundle on X, i.e, for any $b \in B$, $H_{|X_b}$ is an ample line bundle on X_b.

For $k \gg 0$, $L_k = H \otimes (\pi^* L)^k$ defines an ample line bundle on X.

Theorem 2.1 ([ST22, Theorem 1.1])**.** *Suppose that $\mathscr{E} \longrightarrow B$ is a holomorphic vector bundle admitting a HYM connection with respect to $\omega_B \in c_1(L)$. Then for any $\omega_X \in c_1(H)$, there are connections A_k on $\pi^*\mathscr{E}$ which are HYM with respect to $\omega_X + k\pi^*\omega_B$ for all $k \gg 0$.*

If $\mathscr{E}$ is simple and strictly semistable (i.e semistable but not stable) with respect to L, there is a Jordan-Hölder filtration

$$0 \subseteq \mathscr{E}_0 \subseteq \mathscr{E}_1 \subseteq \ldots \subseteq \mathscr{E}_\ell = \mathscr{E}$$

by semistable subsheaves with stable quotient $\mathscr{G}_i = \mathscr{E}_i / \mathscr{E}_{i-1}$ for $i = 1, \ldots, \ell$. We set

$$\mathrm{Gr}(E) := \bigoplus_{i=1}^{\ell} \mathscr{G}_i \qquad \text{and} \qquad \mathfrak{E} := \left\{ \bigoplus_{i \in I} \mathscr{G}_i : \varnothing \subsetneq I \subsetneq \{1, \ldots, \ell\} \right\}.$$

Remark 2.2. By definition of $\mathscr{E}$, there is no HYM connection with respect to any $\omega_B \in c_1(L)$.

If $\mathrm{Gr}(\mathscr{E})$ is locally free, according to the sign of the leading order term in the k-expansion of $\mu_{L_k}(\pi^*\mathscr{E}) - \mu_{L_k}(\pi^*\mathscr{F})$ for $\mathscr{F} \in \mathfrak{E}$, Sektnan-Tipler in [ST22, Theorem 1.4] gave a condition for the existence of HYM connection A_k on $\pi^*\mathscr{E}$ with respect to $\omega_X + k\pi^*\omega_B$ for $k \gg 0$ where $\omega_B \in c_1(L)$ and $\omega_X \in c_1(H)$.

In the paper [NT22], we studied how behave stability when we consider singular varieties and reflexive sheaves in the toric case.

3. Equivariant reflexive sheaves on toric varieties

3.1. **Toric varieties and divisors.** An n-dimensional toric variety is an irreducible variety X containing a torus $T \simeq (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T on itself by multiplication extends to an algebraic action of T on X.

Example 3.1.

(1) $X = \mathbb{C}^n$ with $T = (\mathbb{C}^*)^n$,
(2) $X = \mathbb{P}^n$ with $T \cong (\mathbb{C}^*)^n$,
(3) $X = \{(x, y) \in \mathbb{C}^2 : x^3 - y^2 = 0\}$ with $T = \{(t^2, t^3) : t \in \mathbb{C}^*\}$.

Let N be a rank n lattice and M be its dual with pairing $\langle \cdot, \cdot \rangle : M \times N \longrightarrow \mathbb{Z}$. Then N is the lattice of one-parameter subgroups of the n-dimensional complex torus $T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^*$. Note that M is the character lattice of T_N. We denote by $\chi^m : T_N \longrightarrow \mathbb{C}^*$ the character corresponding to $m \in M$ and we set $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

A *fan* Σ in $N_{\mathbb{R}}$ is a set of rational strongly convex polyhedral cones in $N_{\mathbb{R}}$ such that:

- Each face of a cone in Σ is also a cone in Σ;
- The intersection of two cones in Σ is a face of each.

We will denote $\tau \preceq \sigma$ the inclusion of a face τ in $\sigma \in \Sigma$. A cone σ in $N_{\mathbb{R}}$ is *smooth* if its minimal generators form part of a $\mathbb{Z}$-basis of N. We say that σ is *simplicial* if its minimal generators are linearly independent over $\mathbb{R}$. A fan Σ is *smooth* (resp. *simplicial*) if every cone σ in Σ is smooth (resp. *simplicial*).

For $\sigma \in \Sigma$, let $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ where $\mathbb{C}[S_\sigma]$ is the semi-group algebra of

$$S_\sigma = \sigma^\vee \cap M = \{m \in M \ : \ \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\} \ .$$

If $\sigma, \sigma' \in \Sigma$, we have $U_\sigma \cap U_{\sigma'} = U_{\sigma \cap \sigma'}$. We denote by X_Σ the toric variety associated to a fan Σ; X_Σ is obtained by gluing the affine charts $(U_\sigma)_{\sigma \in \Sigma}$. The variety X_Σ is normal and its torus is $T = N \otimes_{\mathbb{Z}} \mathbb{C}^*$. In general, every normal toric variety comes from a fan [CLS11, Corollary 3.1.8].

By [CLS11, Theorem 3.1.19], the toric variety X_Σ is *smooth* (resp. *$\mathbb{Q}$-factorial*) if and only if the fan Σ is smooth (resp. simplicial).

Let X be the toric variety associated to a fan Σ in $N_{\mathbb{R}}$. For any $\sigma \in \Sigma$, there is a point $\gamma_\sigma \in U_\sigma$ called the *distinguished point* of σ such that the torus orbit $O(\sigma)$ corresponding to σ is given by $O(\sigma) = T \cdot \gamma_\sigma$. By the Orbit-Cone-Correspondence [CLS11, Theorem 3.2.6], there is a bijective correspondence

$$\begin{array}{ccc} \{\text{Cones } \sigma \text{ in } \Sigma\} & \longleftrightarrow & \{T-\text{orbits in } X\} \\ \sigma & \longleftrightarrow & O(\sigma) \end{array}$$

with $\dim O(\sigma) = \dim N_{\mathbb{R}} - \dim \sigma$.

Notation 3.2. We denote by $\Sigma(1)$ the set of one-dimensional cones of Σ and for any $\sigma \in \Sigma$, we set $\sigma(1) = \Sigma(1) \cap \{\tau \in \Sigma : \tau \preceq \sigma\}$. For $\rho \in \Sigma(1)$, we denote by

- $u_\rho \in N$ the minimal generator of ρ;
- D_ρ the closure in the Zariski topology of $O(\rho)$.

So, D_ρ defines an irreducible invariant divisor of X.

3.2. **Equivariant reflexive sheaves.** Let X be a smooth toric variety associated to a fan Σ in $N_{\mathbb{R}}$. Recall that a reflexive sheaf on X is a coherent sheaf $\mathscr{E}$ that is canonically isomorphic to its double dual $\mathscr{E}^{\vee\vee}$.

Let $\theta : T \times X \longrightarrow X$ be the action of T on X, $\mu : T \times T \longrightarrow T$ the group multiplication, $p_2 : T \times X \longrightarrow X$ the projection onto the second factor and $p_{23} : T \times T \times X \longrightarrow T \times X$ the projection onto the second and the third factor. We call a sheaf $\mathscr{E}$ on X *equivariant* if it is equipped with an isomorphism $\Phi : \theta^*\mathscr{E} \to p_2^*\mathscr{E}$ such that

$$(\mu \times \mathrm{Id}_X)^*\Phi = p_{23}^*\Phi \circ (\mathrm{Id}_T \times \theta)^*\Phi \ .$$

Perling [Per04] gave a description of torus equivariant reflexive sheaves over toric varieties in terms of combinatorial data:

Definition 3.3. A family of filtrations $\mathbb{E}$ is the data of a finite dimensional vector space E and for each ray $\rho \in \Sigma(1)$, an increasing filtration $(E^\rho(i))_{i \in \mathbb{Z}}$ of E such that $E^\rho(i) = \{0\}$ for $i \ll 0$ and $E^\rho(i) = E$ for some i.

To a family of filtrations $\mathbb{E} := \left(E, \{E^\rho(j)\}_{\rho \in \Sigma(1), j \in \mathbb{Z}}\right)$, we can assign an equivariant reflexive sheaf $\mathscr{E} := \mathfrak{K}(\mathbb{E})$. The morphisms between families of filtrations are linear maps preserving the filtrations. Then, by [Per04, Theorem 5.18], the functor $\mathfrak{K}$ induces an equivalence of categories between the families of filtrations and equivariant reflexive sheaves over X.

Remark 3.4. The vector space E can be seen as the fiber $\mathscr{E}(x_0)$ where x_0 is the identity element of T and we define the vector subspaces $\{E^\rho(j)\}$ as follows: let $\gamma_\rho \in O(\rho)$ be the distinguished point, we set

$$E^\rho(j) = \left\{ e \in E : \lim_{t\cdot x_0 \to \gamma_\rho, t\in T} \chi^m(t)(t\cdot e) \text{ exists} \right\}$$

where $t \cdot e$ is an element of $\mathscr{E}(t\cdot x_0)$ and $m \in M$ satisfies $\langle m, u_\rho \rangle = j$.

3.3. **Stability of equivariant reflexive sheaves.** Let $\mathscr{E}$ be an equivariant reflexive sheaf on a normal toric variety X given by the family of filtrations $\left(E, \{E^\rho(j)\}_{\rho\in\Sigma(1), j\in\mathbb{Z}}\right)$. By [Koo11, Corollary 3.18], the first Chern class of $\mathscr{E}$ is given by

$$c_1(\mathscr{E}) = -\sum_{\rho\in\Sigma(1)} \iota_\rho(\mathscr{E})\, D_\rho \quad \text{where} \quad \iota_\rho(\mathscr{E}) = \sum_{j\in\mathbb{Z}} j\,(\dim(E^\rho(j)) - \dim(E^\rho(j-1))).$$

Therefore, for any polarization $L \in \mathrm{Amp}(X)$,

$$\mu_L(\mathscr{E}) = -\frac{1}{\mathrm{rk}(\mathscr{E})} \sum_{\rho\in\Sigma(1)} \iota_\rho(\mathscr{E}) \deg_L(D_\rho).$$

According to [Koo11, Proposition 4.13] and [HNS22, Proposition 2.3], to study the stability of $\mathscr{E}$, it is enough to test slope inequalities for equivariant and reflexive saturated subsheaves. These subsheaves are described by families of filtrations $(F, \{E^\rho(j)\cap F\})$ where F is a vector subspace of E.

Notation 3.5. We denote by $\mathscr{E}_F$ the subsheaf of $\mathscr{E}$ given by $(F, \{E^\rho(j)\cap F\})$.

Lemma 3.6. *The set* $\{\mu_L(\mathscr{E}_F) : F \subseteq E \text{ with } 0 < \dim F < \dim E\}$ *is finite.*

4. Pullbacks along fibrations

Let $\pi : X' \longrightarrow X$ be a fibration between $\mathbb{Q}$-factorial projective toric varieties. Let L be an ample line bundle on X and L' a relatively ample line bundle on X'. For $\varepsilon \in \mathbb{Q}_{>0}$ small, $L_\varepsilon = \pi^*L + \varepsilon L'$ define a $\mathbb{Q}$-ample divisor on X'. For a reflexive sheaf $\mathscr{E}$ on X, we denote by $\mathscr{E}' := (\pi^*\mathscr{E})^{\vee\vee}$ its reflexive pullback on X'.

Let Σ and Σ' be respectively the fans of X and X'. If $\mathscr{E}$ is given by $\left(E, \{E^\rho(j)\}_{\rho\in\Sigma(1), j\in\mathbb{Z}}\right)$, then $\mathscr{E}'$ is given by $\left(\widetilde{E}, \{\widetilde{E}^{\rho'}(j)\}_{\rho'\in\Sigma'(1), j\in\mathbb{Z}}\right)$ where $\widetilde{E} = E$ (cf. [NT22, Section 3.1]).

If F is a vector subspace of E, we have

$$\mu_{L_\varepsilon}(\mathscr{E}') - \mu_{L_\varepsilon}(\mathscr{E}'_F) = \left(\mu_{L_\varepsilon}(\mathscr{E}') - \mu_{L_\varepsilon}((\pi^*\mathscr{E}_F)^{\vee\vee})\right) + \left(\mu_{L_\varepsilon}((\pi^*\mathscr{E}_F)^{\vee\vee}) - \mu_{L_\varepsilon}(\mathscr{E}'_F)\right) \tag{1}$$

where

$$\begin{cases} \mu_{L_\varepsilon}((\pi^*\mathscr{E}_F)^{\vee\vee}) - \mu_{L_\varepsilon}(\mathscr{E}'_F) = o(\varepsilon^r) \\ \mu_{L_\varepsilon}(\mathscr{E}') - \mu_{L_\varepsilon}((\pi^*\mathscr{E}_F)^{\vee\vee}) = C(\mu_L(\mathscr{E}) - \mu_L(\mathscr{E}_F))\varepsilon^r + o(\varepsilon^r) \end{cases}$$

with $C > 0$. Therefore, according to Lemma 3.6, it is straightforward to show that: if $\mathscr{E}$ is stable (resp. unstable) with respect to L, then there is $\varepsilon_0 \in \mathbb{Q}_{>0}$ such that for all $\varepsilon \in (0, \varepsilon_0) \cap \mathbb{Q}$, $\mathscr{E}'$ is stable (resp. unstable) with respect to L_ε.

We now consider the case where $\mathscr{E}$ is an equivariant strictly semistable sheaf on (X, L). There is a Jordan-Hölder filtration

$$0 = \mathscr{E}_1 \subseteq \mathscr{E}_2 \subseteq \ldots \subseteq \mathscr{E}_\ell = \mathscr{E}$$

by slope semistable subsheaves with stable quotients of same slope as $\mathscr{E}$. We denote by $\mathrm{Gr}(\mathscr{E}) := \bigoplus_{i=1}^{\ell-1} \mathscr{E}_{i+1}/\mathscr{E}_i$ the graded object of $\mathscr{E}$ and $\mathfrak{E}$ the set of equivariant and saturated reflexive subsheaves $\mathscr{F} \subseteq \mathscr{E}$ arising in a Jordan-Hölder filtration of $\mathscr{E}$. Recall that a locally free semistable sheaf is called *sufficiently smooth* if its graded object is locally free.

For two coherent sheaves $\mathscr{F}_1$ and $\mathscr{F}_2$ on X', we will write $\mu_0(\mathscr{F}_1) < \mu_0(\mathscr{F}_2)$ (resp. $\mu_0(\mathscr{F}_1) \leq \mu_0(\mathscr{F}_2)$ or $\mu_0(\mathscr{F}_1) = \mu_0(\mathscr{F}_2)$) when the coefficient of the smallest exponent in the expansion in ε of $\mu_{L_\varepsilon}(\mathscr{F}_2) - \mu_{L_\varepsilon}(\mathscr{F}_1)$ is strictly positive (resp. greater or equal to zero or equal to zero).

Theorem 4.1 ([NT22, Theorem 1.3]). *Let $\mathscr{E}$ be an equivariant locally free and sufficiently smooth strictly semistable sheaf on (X, L). Then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0) \cap \mathbb{Q}$, the reflexive pullback $\mathscr{E}' := (\pi^*\mathscr{E})^{\vee\vee}$ on (X', L_ε) is:*

(1) stable iff for all $\mathscr{F} \in \mathfrak{E}$, $\mu_0((\pi^*\mathscr{F})^{\vee\vee}) < \mu_0(\mathscr{E}')$,
(2) strictly semistable iff for all $\mathscr{F} \in \mathfrak{E}$, $\mu_0((\pi^*\mathscr{F})^{\vee\vee}) \leq \mu_0(\mathscr{E}')$ with at least one equality,
(3) unstable iff there is one $\mathscr{F} \in \mathfrak{E}$ with $\mu_0((\pi^*\mathscr{F})^{\vee\vee}) > \mu_0(\mathscr{E}')$.

Remark 4.2. In Theorem 4.1 we made the assumptions $\mathscr{E}$ and $\mathrm{Gr}(\mathscr{E})$ are locally free in order to have

$$(\pi^*\mathscr{E}_F)^{\vee\vee} = \mathscr{E}'_F \tag{2}$$

when F satisfies $\mu_L(\mathscr{E}_F) = \mu_L(\mathscr{E})$. In the case where π is a locally trivial fibration, then the assumptions on $\mathscr{E}$ and $\mathrm{Gr}(\mathscr{E})$ to be locally free in Theorem 4.1 are not necessary because (2) is true for any vector subspace $F \subseteq E$.

5. Pullbacks along blowups

Let X be an n-dimensional smooth projective variety with $n \geq 3$ and let Z be a smooth irreducible subvariety of dimension ℓ with $1 \leq \ell \leq n-2$. We denote by $\pi : X' \longrightarrow X$ the blowup of X along Z. Let $\mathscr{E}$ be a reflexive sheaf on X, we have:

$$\mu_{L_\varepsilon}((\pi^*\mathscr{E})^{\vee\vee}) = \mu_L(\mathscr{E}) - \binom{n-1}{\ell-1}\mu_{L_{|Z}}(\mathscr{E}_{|Z})\varepsilon^{n-\ell} + o(\varepsilon^{n-\ell}). \tag{3}$$

Theorem 5.1 ([NT22, Theorem 1.12]). *Let (X, L) be a smooth polarised toric variety. Let $\pi : X' \longrightarrow X$ be the blowup along an invariant irreducible subvariety $Z \subseteq X$ of dimension ℓ with $1 \leq \ell \leq n-2$. We set $L_\varepsilon = \pi^*L - \varepsilon E$ where E is the exceptional divisor of π. Let $\mathscr{E}$ be an equivariant reflexive sheaf that is strictly semistable on (X, L). We assume that for all $\mathscr{F} \in \mathfrak{E}$, $(\pi^*\mathscr{F})^{\vee\vee}$ is saturated in $\mathscr{E}' := (\pi^*\mathscr{E})^{\vee\vee}$ and that*

$$\mu_{L_{|Z}}(\mathscr{E}_{|Z}) < \mu_{L_{|Z}}(\mathscr{F}_{|Z}),$$

then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0) \cap \mathbb{Q}$, the pullback $\mathscr{E}'$ is stable on (X', L_ε).

Example 5.2. Let $X = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1))$ and $\mathrm{pr} : X \longrightarrow \mathbb{P}^1$ be the projection to the base $\mathbb{P}^1$. The tangent sheaf $\mathscr{T}_X$ is strictly semistable with respect to $L = \mathrm{pr}^*\,\mathscr{O}_{\mathbb{P}^1}(1) \otimes \mathscr{O}_X(3)$.

Let C_1, C_2 be two invariant curves (by the action of the torus of X) such that

$$\mathrm{pr}(C_1) = \{pt\} \qquad \text{and} \qquad \mathrm{pr}(C_2) = \mathbb{P}^1.$$

For $i \in \{1, 2\}$, we set $\pi_i : X_i = \mathrm{Bl}_{C_i}(X) \longrightarrow X$ and $\mathscr{E}_i = (\pi^*\mathscr{T}_X)^{\vee\vee}$. We denote by D_i the exceptional divisor of π_i. We have:

(1) There is $\varepsilon_0 > 0$ such that $\mathscr{E}_1$ is stable with respect to $\pi^*L - \varepsilon D_1$ for any $\varepsilon \in (0, \varepsilon_0) \cap \mathbb{Q}$.
(2) There is $\varepsilon_0 > 0$ such that $\mathscr{E}_2$ is unstable with respect to $\pi^*L - \varepsilon D_2$ for any $\varepsilon \in (0, \varepsilon_0) \cap \mathbb{Q}$.

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