On the harmonic flow of geometric structures

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UFRJ

Workshop BRIDGES

June 21, 2023

H-structures

Let M^n be a connected and orientable smooth n-manifold without boundary.

- $\mathrm{Fr}(M) := \bigcup_{x\in M}\{u : T_xM \to \mathbb{R}^n | u \text{ linear isomorphism}\}$: principal $\mathrm{GL}(n,\mathbb{R})$-bundle.
- $H \subset \mathrm{GL}(n,\mathbb{R})$ Lie subgroup; an H-**structure** on M is a principal H-subbundle $Q \subset \mathrm{Fr}(M)$. (purely topological)
- *e.g.*: $\mathrm{SO}(n)$-structure Q on $M \iff$ Riemannian metric g and orientation on M; Q is the principal $\mathrm{SO}(n)$-bundle of oriented orthonormal coframes of (M,g), which we will write as $\pi_{\mathrm{SO}(n)} : \mathrm{Fr}(M,g) \to M$.
- Assume from now on $H \subset \mathrm{SO}(n)$ closed and connected. Then, any H-structure Q induces a unique $\mathrm{SO}(n)$-structure P such that $Q \subset P$ ($P = \mathrm{SO}(n)\cdot Q$).
- Nonetheless, there are many H-structures inducing the same $\mathrm{SO}(n)$-structure $\mathrm{Fr}(M,g)$; note that $H \curvearrowright \mathrm{Fr}(M,g)$ freely and quotient map $\pi_H : \mathrm{Fr}(M,g) \to \mathrm{Fr}(M,g)/H$ is a principal H-bundle.
- $\rightsquigarrow \pi : \mathrm{Fr}(M,g)/H \to M$ such that $\pi_{\mathrm{SO}(n)} = \pi \circ \pi_H$ is a fiber bundle $\cong \mathrm{Fr}(M,g) \times_{\mathrm{SO}(n)} \mathrm{SO}(n)/H$.
- There is a bijection:

$$\{\text{Compatible } H\text{-structures on } (M,g)\} \longleftrightarrow \Gamma(\mathrm{Fr}(M,g)/H)$$

$$Q \mapsto (\sigma_Q : x \mapsto \pi_H(u), u \in \pi^{-1}_{\mathrm{SO}(n)}(x) \cap Q)$$

$$Q_\sigma := \pi_H^{-1}(\sigma(M)) \leftarrow\!\shortmid \sigma$$

- $H \subset \mathrm{SO}(n)$ closed and connected $\Rightarrow \mathrm{SO}(n)/H$ is a normal homogeneous Riemannian n-manifold with canonical bi-invariant metric $\langle A, B\rangle := -\mathrm{tr}(AB)$.

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Intrinsic torsion of a H-structure

- H-module decomposition $\mathfrak{so}(n) = \mathfrak{h} \oplus \mathfrak{m}$, with $\mathfrak{m} := \mathfrak{h}^\perp$ with respect to $\langle\cdot,\cdot\rangle$, is *reductive*: $\mathrm{Ad}_{\mathrm{SO}(n)}(H)\mathfrak{m} \subseteq \mathfrak{m}$.
- $Q \subset \mathrm{Fr}(M,g)$ compatible H-structure $\rightsquigarrow$ H-module decomposition of the bundle $\mathfrak{so}(TM) := \mathrm{Fr}(M,g) \times_{\mathrm{SO}(n)} \mathfrak{so}(n) \subseteq \mathrm{End}(TM) = T^*M \otimes TM$:

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- A connection $\tilde\nabla$ on TM is **compatible with the H-structure Q** (**H-connection**), if the connection 1-form $\tilde\omega \in \Omega^1(\mathrm{Fr}(M), \mathfrak{gl}(n,\mathbb{R}))$ on $\mathrm{Fr}(M)$ *reduces* to Q. These are precisely the ones induced by connections on Q. Since Q is compatible with g, any H-connection $\tilde\nabla$ on TM preserves g, and denoting by ∇ the Levi–Civita connection of (M^n,g), it follows that $\tilde T_X := \tilde\nabla_X - \nabla_X \in \Gamma(\mathfrak{so}(TM))$, for all $X \in \mathfrak{X}(M)$. Essentially, $\tilde T$ is the *torsion* of $\tilde\nabla$. Writing $\tilde T_X = \pi_{\mathfrak{h}}(\tilde T_X) + \pi_{\mathfrak{m}}(\tilde T_X)$, we can define the H-connection $\nabla^H_X := \tilde\nabla_X - \pi_{\mathfrak{h}}(T_X)$. Since the difference between any two H-connections lies in $\Gamma(\mathfrak{h}_Q)$, it follows that ∇^H is the unique H-connection on M the torsion $T = T^Q$ of which satisfies

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$T \in \Omega^1(M, \mathfrak{m}_Q)$ is called the **intrinsic torsion** of Q, and Q is said to be *torsion-free* when $T \equiv 0$ ($\iff$ ∇ is an H-connection, $\mathrm{Hol}(g) \subset H$).

Geometric structures: H-structures characterized by their stabilized tensors

- Given an H-structure $Q \subset \mathrm{Fr}(M)$, a tensor field $\xi \in \Gamma(\mathcal{T}^{p,q}(TM))$ is said to be *stabilized by H* if, for any adapted H-coframe $u \in Q$ one has $H \subseteq \mathrm{Stab}(u^{-1}.\xi) \subseteq \mathrm{GL}(n, \mathbb{R})$.
- In what follows, we shall be interested in H-structures that are completely characterized by their stabilized tensors. This amounts to assuming that the group H is the **stabilizer of one or several tensors** on $\mathbb{R}^n$, meaning $H = \mathrm{Stab}(\xi_\circ)$ for some element $\xi_\circ = ((\xi_\circ)_1, \dots, (\xi_\circ)_k)$ in a subspace $V \leqslant \oplus \mathcal{T}^{p,q}(\mathbb{R}^n)$, $V = V_1 \oplus \dots \oplus V_k$ with $V_i \leqslant \mathcal{T}^{p_i,q_i}(\mathbb{R}^n)$. Then Q corresponds bijectively to a **geometric structure** ξ modelled on $\xi_\circ$: for each $x \in M$, there exists a coframe $u : T_x M \to \mathbb{R}^n$ identifying $\xi(x)$ and $\xi_\circ$.
- e.g.: for $H = U(m) \subset \mathrm{SO}(2m)$ we can take $\xi_\circ = (g_\circ, J_\circ) \in \Sigma^2 \oplus \mathrm{End}(\mathbb{R}^{2m})$, where $g_\circ$ and $J_\circ$ are the standard flat metric and complex structure, respectively; for $H = \mathrm{G}_2 \subset \mathrm{SO}(7)$, we can take $\xi_\circ = \varphi_\circ \in \Omega^3(\mathbb{R}^7)$ the standard positive 3-form, etc.

Infinitesimal deformations

The canonical right $\mathrm{GL}(n,\mathbb{R})$-action on tensors induces an **infinitesimal action** of endomorphisms $A \in \Gamma(\mathrm{End}(TM))$ on elements $\xi \in \Gamma(\mathcal{T}^{p,q}(TM))$ given by

$$A \diamond \xi := \frac{d}{dt}\Big|_{t=0} e^{tA}.\xi.$$

Using the metric g, we let $A_{ij} := g_{lj}A_i^l$ and we decompose $A = S + C \in \Sigma^2(M) \oplus \Omega^2(M)$, where $S_{ij} = \frac{1}{2}(A_{ij} + A_{ji})$ and $C_{ij} = \frac{1}{2}(A_{ij} - A_{ji})$.

Lemma

For all $A, B \in \Gamma(\mathrm{End}(TM))$ and $\xi \in \Gamma(\mathcal{T}^{p,q}(TM))$, the ***diamond operator*** *$\diamond$ satisfies:*

(i) $A \diamond B = -[A, B]$.

(ii) $A \diamond (B \diamond \xi) - B \diamond (A \diamond \xi) = -[A, B] \diamond \xi$.

(iii) *If $\xi \in \Gamma(\mathcal{T}^{0,q}(TM))$ is a symmetric (resp. alternating) tensor, then so is $A \diamond \xi$.*

(iv) $g \diamond \xi = (q - p)\xi$.

(v) $A = S + C \in \Sigma^2(M) \oplus \Omega^2(M) \Rightarrow A \diamond g = 2S$; *in particular,* $\ker(\cdot \diamond g) = \Omega^2$.

(vi) $A \diamond \mathrm{vol}_g = \mathrm{tr}(A)\mathrm{vol}_g$; *in particular,* $\ker(\cdot \diamond \mathrm{vol}_g) = \Sigma_0^2 \oplus \Omega^2$.

(vii) *If $D \in \Omega^2(M)$ then $\langle D \diamond \xi, \xi \rangle_g = 0$.*

(viii) *If $D \in \Omega^2(M)$ then $\langle A \diamond \xi, D \diamond \xi \rangle_g = -\langle D \diamond (A \diamond \xi), \xi \rangle_g$.*

The diamond operator and compatible H-structures

Now suppose $Q \subset \mathrm{Fr}(M,g)$ is a compatible H-structure. Then we get a corresponding H-module decomposition on $\Lambda^2(T^*M) \simeq \mathfrak{so}(TM)$:

$$\Lambda^2 = \Lambda^2_{\mathfrak{h}} \oplus \Lambda^2_{\mathfrak{m}}, \quad \Lambda^2_{\mathfrak{h}} \simeq \mathfrak{h}_Q \quad \text{and} \quad \Lambda^2_{\mathfrak{m}} \simeq \mathfrak{m}_Q.$$

We shall write $\Omega^2_{\mathfrak{h}} := \Gamma(\Lambda^2_{\mathfrak{h}})$ and $\Omega^2_{\mathfrak{m}} := \Gamma(\Lambda^2_{\mathfrak{m}})$. Then, splitting out the trivial submodule Ω^0 of $\Sigma^2(M)$ spanned by the Riemannian metric g, we have

$$\Gamma(\mathrm{End}(TM)) \simeq \Omega^0 \oplus \Sigma^2_0 \oplus \Omega^2_{\mathfrak{h}} \oplus \Omega^2_{\mathfrak{m}}.$$

Lemma

The following hold:

(i) *If $\xi \in \Gamma(\mathcal{T}^{p,q}(TM))$ is stabilized under the action of H, then $\Omega^2_{\mathfrak{h}} \subseteq \ker(\cdot \diamond \xi)$.*

(ii) *If $H = \mathrm{Stab}(\xi_\circ)$, so that Q corresponds to a geometric structure $\xi = (\xi_1, \dots, \xi_k)$ modelled on $\xi_\circ$, then*

$$\Omega^2_{\mathfrak{h}} = \ker(\cdot \diamond \xi) = \ker(\cdot \diamond \xi_1) \cap \dots \cap \ker(\cdot \diamond \xi_k).$$

(iii) *If $H = \mathrm{Stab}_{\mathrm{SO}(n)}(\xi_\circ)$, then*

$$\Omega^2_{\mathfrak{h}} = \ker(\cdot \diamond \xi) \cap \Omega^2.$$

Intrinsic torsion and diamond operator

Lemma

Let $Q \subset \mathrm{Fr}(M, g)$ be a compatible H-structure. If $\xi \in \Gamma(\mathcal{T}^{p,q}(TM))$ is stabilized under the action of H, then

$$\nabla_X \xi = T_X \diamond \xi, \quad \forall X \in \mathfrak{X}(M), \tag{1.4}$$

In particular, if $H = \mathrm{Stab}_{\mathrm{SO}(n)}(\xi_\circ)$ and Q is thus determined by a geometric structure ξ modelled on $\xi_\circ$, then there are constants $c, \tilde{c} > 0$, depending only on (M, g) and H, such that

$$\tilde{c}|T|^2 \leqslant |\nabla\xi|^2 \leqslant c|T|^2. \tag{1.5}$$

If furthermore there is $c > 0$ such that $\langle A \diamond \xi, B \diamond \xi \rangle = c\langle A, B\rangle$, for all $A, B \in \Omega^2_{\mathfrak{m}}(M)$ (e.g. if $\mathfrak{m}$ is an irreducible H-module), then in fact

$$|\nabla\xi|^2 = c|T|^2. \tag{1.6}$$

Inner product relations

Lemma

Suppose $H = \mathrm{Stab}_{\mathrm{SO}(n)}(\xi_\circ)$ *and* ξ *is a compatible* H*-structure on* (M, g)*. Let* $\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_k$ *be an orthogonal decomposition of* $\mathfrak{m}$*, with respect to the bi-invariant metric* $\langle A, B\rangle = -\mathrm{tr}(AB)$*, into non-equivalent, irreducible* $\mathrm{Ad}_{\mathrm{SO}(n)}(H)$*-submodules. Then:*

(i) $\exists \lambda_i \in \mathbb{R}_+$ *such that, for all* $A, B \in \Omega^2_{\mathfrak{m}}(M)$,

$$\langle A \diamond \xi, B \diamond \xi\rangle = \sum_{i=1}^{k} \lambda_i \langle A_i, B_i\rangle,$$

where $A_i := \pi_{\mathfrak{m}_i}(A)$*,* $B_i := \pi_{\mathfrak{m}_i}(B)$*, for* $i = 1, \ldots, k$*.*

(ii) *In particular,*

$$\langle C \diamond (C \diamond \xi), D \diamond \xi\rangle = \sum_{i=1}^{k} \lambda_i \langle [C, D], C_i\rangle, \quad \forall C, D \in \Omega^2_{\mathfrak{m}}(M), \tag{1.8}$$

and this equals zero if $\lambda_1 = \ldots = \lambda_k$ *(e.g. when* $\mathfrak{m}$ *is irreducible).*

Example: $\mathrm{U}(m)$ case

Consider the case where $H = \mathrm{U}(m) = \mathrm{Stab}_{\mathrm{SO}(2m)}(J_\circ) \subset \mathrm{SO}(2m)$. Then $\mathfrak{m} = \mathfrak{u}(m)^\perp = \{A \in \mathfrak{so}(n) : AJ_\circ = -J_\circ A\}$ is irreducible, and for any compatible $\mathrm{U}(m)$-structure $\xi = J$ on (M^{2m}, g), we can compute, for all $A, B \in \Omega^2_{\mathfrak{m}}(M)$,

$$\begin{aligned}\langle A \diamond J, B \diamond J\rangle &= \langle [A, J], [B, J]\rangle = \langle 2AJ, (-2)JB\rangle = 4\mathrm{tr}(AJJB) \\ &= 4\langle A, B\rangle.\end{aligned}$$

Moreover,

$$\nabla_X J = (T_X \diamond J) = -[T_X, J] = 2JT_X, \quad \forall X \in \mathfrak{X}(M),$$

since $T_X \in \Omega^2_{\mathfrak{u}(m)^\perp} \simeq \{A \in \mathfrak{so}(M) : AJ = -JA\}$. Thus,

$$T_X = -\frac{1}{2} J\nabla_X J, \quad \forall X \in \mathfrak{X}(M).$$

In particular,

$$|\nabla J|^2 = 4|T|^2.$$

Moreover, $\cdot \diamond J$ maps $\Omega_{\mathfrak{u}(m)^\perp}$ into itself, so that $\nabla_X J \in \Omega^2_{\mathfrak{u}(m)^\perp}$, $\forall X \in \mathfrak{X}(M)$.

Rough Laplacian and the diamond operator

Write $\Delta := -\nabla^*\nabla$, so that at the center of normal coordinates $\Delta = \nabla_k \nabla_k$.

Lemma

Suppose $Q \subset \mathrm{Fr}(M, g)$ is a compatible H-structure with torsion T. If $\xi \in \Gamma(\mathcal{T}^{p,q}(TM))$ is stabilized by H then

$$\Delta\xi = \mathrm{div}_g T \diamond \xi + T_k \diamond (T_k \diamond \xi), \tag{1.10}$$

where $(\mathrm{div}_g T)_{ij} := \nabla_k T_{k;ij} \in \Omega^2_{\mathfrak{m}}(M)$.
In particular, if $H = \mathrm{Stab}_{\mathrm{SO}(n)}(\xi_\circ)$ and Q is then determined by a geometric structure ξ modelled on $\xi_\circ$, then $\exists c > 0$, depending only on (M, g) and H, such that if $\mathrm{div}_g T = 0$ then

$$|\Delta\xi| \leqslant c|\nabla\xi|^2. \tag{1.11}$$

If furthermore there is $c > 0$ such that $\langle A \diamond \xi, B \diamond \xi \rangle = c\langle A, B \rangle$ for all $A, B \in \Omega^2_{\mathfrak{m}}(M)$, i.e. if $c := \lambda_1 = \ldots = \lambda_k$ (e.g. if $\mathfrak{m}$ is an irreducible H-module), then the decomposition (1.10) of $\Delta\xi$ is orthogonal.

General flows of H-structures

When M admits a H-structure ξ defined by one or several tensor fields which are stabilized by $H \subset \mathrm{SO}(n)$, we saw in particular that $\Omega^2_{\mathfrak{h}}$ is a subspace of $\ker(\cdot \diamond \xi)$. Consequently, a general $\mathrm{GL}(n, \mathbb{R})$-variation of ξ can be written as:

$$\frac{\partial}{\partial t}\xi = A \diamond \xi, \quad A \equiv A(t) = S(t) + C(t), \quad S(t) \in \Sigma^2, C(t) \in \Omega^2_{\mathfrak{m}} \subset \Omega^2. \qquad (1.12)$$

Moreover, if $\{\xi(t)\}_{t \in I \ni 0}$ is a family of H-structures evolving under (1.12), and if $g(t)$ is the unique Riemannian metric on M^n determined by $\xi(t)$, then

$$\frac{\partial}{\partial t} g(t) = A(t) \diamond g(t) = 2S(t).$$

In particular, the flow is isometric iff $S(t) \equiv 0$.

Dirichlet energy functionals

Suppose (M, g) is closed and let ξ be a compatible geometric H-structure. Consider the following energy functionals:

$$\mathcal{E}(\xi) := \frac{1}{2}\int_M |T_\xi|^2 \mathrm{vol}_g \quad \text{and} \quad \mathcal{D}(\xi) := \frac{1}{2}\int_M |\nabla\xi|^2 \mathrm{vol}_g.$$

Then, by previous lemmas, there are $c, \tilde{c} > 0$ depending only on (M, g) and H such that

$$\tilde{c}\mathcal{E}(\xi) \leqslant \mathcal{D}(\xi) \leqslant c\mathcal{E}(\xi).$$

Moreover, under the assumption that $c := \lambda_1 = \ldots = \lambda_k$, i.e. if there is $c > 0$ such that $\langle A \diamond \xi, B \diamond \xi \rangle = c\langle A, B \rangle$ for all $A, B \in \Omega^2_{\mathfrak{m}}(M)$ (e.g. if $\mathfrak{m}$ is an irreducible H-module), then

$$\mathcal{D}(\xi) = c\mathcal{E}(\xi).$$

First variation of the Dirichlet energy

Lemma

Suppose that $H = \mathrm{Stab}_{\mathrm{SO}(n)}(\xi_\circ)$ *is such that* $\lambda_1 = \ldots = \lambda_k$ *(e.g. when* $\mathfrak{m}$ *is an irreducible H-module). If* $\{\xi(t)\}$ *is a smooth family of compatible H-structures on* (M^n, g)*, with* $\xi(0) = \xi$ *and* $\frac{d}{dt}|_{t=0}\xi(t) = C \diamond \xi$*, for some* $C \in \Omega^2_{\mathfrak{m}}$*, then*

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{D}(\xi(t)) = -\int_M \langle C \diamond \xi, \mathrm{div}_g T \diamond \xi \rangle \mathrm{vol}_g.$$

Thus, the energy $\mathcal{D}$ *restricted to compatible H-structures on* (M^n, g) *has gradient* $-\mathrm{div}_g T \diamond \xi$ *at each point* ξ*.*

Harmonic geometric structures

This motivates a natural harmonicity theory:

Definition

A family of compatible H-structures $\{\xi(t)\}_{t\in I}$ on (M,g), parameterised by a non-degenerate interval $I \subset \mathbb{R}$, is a solution to the *harmonic flow of H-structures* (or *harmonic H-flow* for short) if the following evolution equation holds for every $t \in I$:

$$\frac{\partial}{\partial t}\xi(t) = \operatorname{div}_g T(t) \diamond \xi(t), \tag{HF}$$

where $T(t)$ denotes the torsion of $\xi(t)$. Given a compatible H-structure ξ_0 on (M^n, g), a solution to the harmonic flow of H-structures with *initial condition* (or *starting at*) ξ_0 is a solution of (HF) defined for every $t \in [0, \tau_0)$, for some $0 < \tau_0 \leqslant \infty$, and such that $\xi(0) = \xi_0$.

Definition

When ξ is a compatible H-structure on (M^n, g), we say that ξ is *harmonic* when it has divergence-free torsion:

$$\operatorname{div}_g T = 0.$$

Some of the previous literature on harmonic structures

- The problem of fixing a metric g and looking for a "best" compatible H-structure dates back to Calabi–Gluck ($\mathrm{U}(3)$-structures on S^6) and C. Wood in the 1990s; Wood introduced the general notion of harmonicity from the point of view of sections of twistor bundles. In the case $H = U(m) \subset \mathrm{SO}(2m)$, i.e. for an almost complex structure J compatible with (M, g), the equation $\mathrm{div}_g T = 0$ becomes
$$[\nabla^* \nabla J, J] = 0.$$
Of course, Kähler structures are absolute minimizers of the energy $\mathcal{D}$ but $\exists$ various absolute minimizers which are not Kähler (Bor-Lamoneda-Salvai '07).
- González-Dávila and Martín Cabrera (2008) investigates general harmonic H-structures, with a focus on $U(m)$-structures, characterizing harmonicity according to each torsion class; when $m = 3$, $T \in \Lambda^1 \otimes \mathfrak{u}(m)^\perp = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, where $\mathcal{W}_i$ are irreducible $U(m)$-modules given by Gray–Hervella.
- The harmonic flow of G_2-structures was more recently investigated by Grigorian (2017, 2019), Bagaglini (2019), and Dwivedi–Gianniotis–Karigiannis (2019), while the harmonic flow of $U(m)$-structures was studied by He–Li (2019). The general harmonic flow of H-structures was introduced and investigated by Loubeau–Sá Earp (2019).
- More recent works include the $\mathrm{Spin}(7)$ case of the flow by Dwivedi–Loubeau–Sá Earp (2021), the work on harmonic $\mathrm{Sp}(2)$-invariant G_2-structures on S^7 by Loubeau–Moreno–Sá Earp–Saavedra (2022), and the study of the harmonic flow of quaternion-Kähler structures by Fowdar–Sá Earp (2023).

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Short-time existence and uniqueness of the harmonic flow

There is a natural isomorphism between $\pi : \mathrm{Fr}(M,g)/H \to M$ and the associated bundle $\mathrm{Fr}(M,g) \times_{\mathrm{SO}(n)} \mathrm{SO}(n)/H$, which fibrewise is an isometry with respect to the bi-invariant metric on $\mathrm{SO}(n)$. The induced one-to-one correspondence between sections $\sigma \in \Gamma(\mathrm{Fr}(M,g)/H)$ and $\mathrm{SO}(n)$-equivariant maps $s : \mathrm{Fr}(M,g) \to \mathrm{SO}(n)/H$ identifies solutions to the harmonic section flow with $\mathrm{SO}(n)$-equivariant solutions to the classical harmonic map heat flow for maps $\mathrm{Fr}(M,g) \to \mathrm{SO}(n)/H$, where the target space $\mathrm{SO}(n)/H$ is considered with its normal homogeneous Riemannian manifold structure.

Theorem (Loubeau–Sá Earp (2019))

Given any smooth compatible H-structure ξ_0 on (M^n, g), there is a maximal time $\tau = \tau(\xi_0) \in (0, \infty]$ such that the harmonic H-flow with initial condition ξ_0 admits a unique smooth solution $\xi(t)$ for $t \in [0, \tau)$. Moreover, if $\tau < \infty$ then $\sup_M |\nabla \xi(t)| \to \infty$ as $t \to \tau$.

Assumptions for our analytical results on the harmonic flow

- $H = \mathrm{Stab}_{\mathrm{SO}(n)}(\xi_\circ)$, where $\xi_\circ$ is an element of a r-dimensional $\mathrm{SO}(n)$-submodule $V \leqslant \oplus \mathcal{T}^{p,q}(\mathbb{R}^n)$. Furthermore, we assume that H is such that $\lambda_1 = \ldots = \lambda_k$, i.e. there is $c > 0$ such that $\langle A \diamond \xi, B \diamond \xi \rangle = c\langle A, B \rangle, \quad \forall A, B \in \Omega^2_{\mathfrak{m}}(M)$.
- Suppose (M^n, g) admits a compatible H-structure ξ_0 and let $\{\xi(t)\}_{t\in[0,\tau)}$ be a solution to the **harmonic H-flow** on (M, g) with initial condition $\xi(0) = \xi_0$:

$$\partial_t \xi = \mathrm{div}_g T \diamond \xi, \quad \xi(0) = \xi_0.$$

- Let $0 < r_M < \mathrm{inj}(M, g)$. Then, restricted to the geodesic ball $B_{r_M}(y)$ we can regard ξ as a tensor defined on $B_{r_M}(0) \times [0, \tau) \subset \mathbb{R}^n \times [0, \tau)$ via normal coordinates. Fix any $\tau_0 \in (0, \tau)$ and a cut-off function $\phi \in C_c^\infty(B_{r_M}(0))$ with $\phi|_{B_{\frac{r_M}{2}}(0)} \equiv 1$. For all $t \in (0, \tau_0)$ and $0 < r \leqslant \min\{\sqrt{\tau_0}/2, r_M\}$, we define

$$\Psi(r) \equiv \Psi_{(y,\tau_0)}(r) := \int_{\tau_0 - 4r^2}^{\tau_0 - r^2} \int_{\mathbb{R}^n} |\nabla \xi|^2 k_{(0,\tau_0)} \phi^2 \sqrt{\det(g)}\, dx dt.$$

ε-regularity along the flow

Theorem (F-, Loubeau, Moreno, Sá Earp (2022))

There exists a constant $\varepsilon_0 > 0$, depending only on (M^n, g), the group H, and the energy of the initial data such that, if $\Psi_{(y,\tau_0)}(r) < \varepsilon_0$, then

$$\sup_{P_{\delta r}(y,\tau_0)} |\nabla \xi|^2 \leqslant 4(\delta r)^{-2},$$

where $P_{\delta R}$ is a parabolic neighbourhood, and the constant $\delta > 0$ depends only on the geometry and initial data.

The main ingredients to prove this result are:

- **Almost monotonicity formula**: $\forall 0 < R_1 \leqslant R_2 \leqslant \min\{\sqrt{\tau_0}/2, r_M\}$ and $\forall N > 1$,

$$\Psi(R_1) \leqslant C\Psi(R_2) + C\left(N^{n/2}(E_0 + \sqrt{E_0}) + \frac{C}{\ln^2 N}\right)(R_2 - R_1).$$

- **Bochner-type estimate** for $e(\xi) := \frac{1}{2}|\nabla \xi|^2$:

$$(\partial_t - \Delta)\, e(\xi) \leqslant C_H(e(\xi)^2 + 1).$$

An energy gap theorem

Theorem (F-, Loubeau, Moreno, Sá Earp (2022))

Let (M^n, g) be a closed Riemannian n-manifold admitting a compatible H-structure. Then there is a constant $\varepsilon_0(M^n, g, H) > 0$ such that if ξ is a compatible harmonic H-structure on (M^n, g) whose energy satisfies $\mathcal{D}(\xi) := \frac{1}{2}\|\nabla\xi\|^2_{L^2(M)} < \varepsilon_0$, then ξ is actually torsion-free, i.e. $\nabla\xi = 0$.

Proof.

If not true, then $\exists$ sequence $(\xi_k)_{k=1}^{\infty}$ of harmonic H-structures inducing g such that $\mathcal{D}(\xi_k) \to 0$ as $k \to \infty$ but $\nabla\xi_k \neq 0$ for all k. By the ε-**regularity** and **Shi-type estimates**, it follows that for $k \gg 1$ we have that $|\nabla^m \xi_k|$ is uniformly bounded $\forall m \in \mathbb{N}_0$. Therefore, up to taking a subsequence, $\xi_k \to \xi$ in C^∞ and $\nabla\xi = 0$. Since ξ_k is harmonic and $\nabla\xi = 0$, there is a uniform $c > 0$ such that

$$|\Delta(\xi_k - \xi)| \leqslant c|\nabla(\xi_k - \xi)|^2.$$

Combining this with integration by parts gives

$$\int_M |\nabla(\xi_k - \xi)|^2 = -\int_M \langle \xi_k - \xi, \Delta(\xi_k - \xi)\rangle \leqslant c\|\xi_k - \xi\|_{L^\infty(M)} \int_M |\nabla(\xi_k - \xi)|^2.$$

Since $\xi_k \to \xi$ as $k \to \infty$ in smooth topology, we get a contradiction. ■

A finite-time singularity result for the harmonic H-flow

Theorem (F-, Loubeau, Moreno, Sá Earp (2022))

Let (M^n, g) be a closed oriented Riemannian n-manifold endowed with a compatible H-structure $\bar{\xi}$ whose isometric homotopy class $[\bar{\xi}]$ does not contain any torsion-free H-structure but

$$\inf_{\xi \in [\bar{\xi}]} \mathcal{D}(\xi) = 0.$$

Then there is a constant $\varepsilon_*(M, g, H) > 0$ such that if $\xi_0 \in [\bar{\xi}]$ is such that $\mathcal{D}(\xi_0) < \varepsilon_*$, then the harmonic H-flow starting at ξ_0 develops a finite time singularity. Moreover, if $[0, \tau(\xi_0))$ denotes the maximal existence interval for the solution, then $\tau(\xi_0)^{n-2} \lesssim \mathcal{D}(\xi_0)$; in particular, $\tau(\xi_0) \to 0$ as $\mathcal{D}(\xi_0) \to 0$.

Example(s) of finite-time singularity for the harmonic H-flow

- $M := T^7$ endowed with the standard G_2-structure φ_0 inducing the flat metric g_0.
- Frame bundle of (T^7, g_0) is trivial and so is the twistor bundle $\mathrm{Fr}(T^7, g_0)/\mathrm{G}_2 = T^7 \times SO(7)/\mathrm{G}_2 = T^7 \times \mathbb{RP}^7$ of G_2-structures isometric to φ_0.
- $\therefore$ any G_2-structure φ isometric to φ_0 can be thought of as a map from T^7 to $\mathbb{RP}^7$, and under such identification the standard G_2-structure φ_0 corresponds to a constant map.
- Fix r_0 small enough so that the geodesic ball $B(p, r_0) \subset (T^7, g_0)$ is isometric to the Euclidean ball $B(0, r_0) \subset \mathbb{R}^7$, and consider an isometric G_2-structure $\varphi \in [[\varphi_0]]$ which is the constant map φ_0 outside $B(p, r_0)$. In particular, we can think of φ (restricted to $B(p, r_0)$) as a map from the 7-sphere S^7 to $\mathbb{RP}^7$. In this sense, the isometric homotopy class of φ then corresponds to an element of $\pi_7(\mathbb{RP}^7) = \pi_7(S^7) = \mathbb{Z}$, and $\varphi \in [\varphi_0]$ if and only if such an element is the trivial one. Choose φ such that its isometric homotopy class corresponds to any nonzero element in $\pi_7(\mathbb{RP}^7) = \mathbb{Z}$. Up to a deformation within its isometric hotomopy class, we can assume that φ is a smooth G_2-structure, which by construction is isometric to φ_0 but $[\varphi] \neq 0 = [\varphi_0]$.

Example(s) of finite-time singularity for the harmonic H-flow

- $\forall r \in (0, r_0]$, let φ_r be the G_2-structure on T^7 such that $\varphi_r|_{X\setminus B(p,r)} = \varphi_0$, and

$$\varphi_r(x) := \varphi\left(\frac{xr_0}{r}\right), \quad \forall x \in B(p,r) \simeq B(0,r) \subset \mathbb{R}^7.$$

- φ_r is isometric to φ_0 and $\varphi_r \in [\varphi] \neq 0 = [\varphi_0]$. We compute the energy of φ_r:

$$\begin{aligned}
2\mathcal{D}(\varphi_r) &= \int_{B(p,r)} |\nabla\varphi_r|^2(x)dx \quad (\text{since } \varphi_r = \varphi_0 \text{ outside } B(p,r)) \\
&= \int_{B(p,r)} \left|\nabla\left(\varphi\left(\frac{xr_0}{r}\right)\right)\right|^2 dx \quad (\text{by the def of } \varphi_r) \\
&= r_0^2 r^{-2} \int_{B(p,r)} |\nabla\varphi|^2\left(\frac{xr_0}{r}\right) dx \\
&= r_0^2 r^{-2} r_0^{-7} r^7 \int_{B(p,r_0)} |\nabla\varphi|^2(y)dy \quad (\text{by change of variables}) \\
&= r_0^{-5} r^5 \mathcal{D}(\varphi). \quad (\text{since } \varphi = \varphi_0 \text{ outside } B(p,r_0))
\end{aligned}$$

In particular, $\mathcal{D}(\varphi_r) \to 0$ as $r \to 0$ and therefore

$$\inf_{\tilde\varphi\in[\varphi]} \mathcal{D}(\tilde\varphi) = 0.$$

Example(s) of finite-time singularity for the harmonic H-flow

- On the other hand, $[\varphi]$ cannot contain any torsion-free G_2-structure, since any such would correspond to a constant map from T^7 to $\mathbb{RP}^7$ and $[\varphi]$ is non-trivial.
- **Claim**: For $r \ll 1$, the harmonic G_2-flow starting at φ_r has a finite time singularity.
- Indeed: if otherwise the flow $\varphi(t)$ with $\varphi(0) = \varphi_r$ exist for all time $t > 0$ then, since $r \ll 1$, and thus $\mathcal{D}(\varphi_r) \ll 1$, it follows from the ε-regularity together with Shi-type estimates that $\varphi(t)$ converges smoothly as $t \to \infty$ to a divergence-free torsion G_2-structure $\varphi_\infty \in [\varphi_r] = [\varphi]$. In fact, if $r \ll 1$ is small enough, φ_∞ would be torsion-free because of the energy gap and the fact that the energy is non-increasing along the flow. But φ_∞ being torsion-free implies that its homotopy class corresponds to that of a constant map from T^7 to $\mathbb{RP}^7$, contradicting the non-triviality of $[\varphi] \neq 0 = [\varphi_0]$.
- Notice that although on the one hand the φ_r have arbitrarily small energy $\mathcal{D}(\varphi_r) \to 0$ as $r \to 0$, on the other hand the L^∞-norm of its torsion is blowing-up as $r \to 0$:

$$\|\nabla \varphi_r\|_{L^\infty(M)} = r_0 r^{-1} \|\nabla \varphi\|_{L^\infty(B(p,r_0))} \to \infty \quad \text{as } r \to 0.$$

  Thus, this example also illustrates that a general long-time existence result for the harmonic flow under the hypothesis of small initial energy should take into account the L^∞-norm of the initial torsion in the smallness condition.

Long-time existence under small initial energy

Theorem (F-, Loubeau, Moreno, Sá Earp (2022))

Let (M^n, g) be a closed, oriented Riemannian n-manifold admitting a compatible H-structure. Then for any given constant $K > 0$, there is a universal constant $\varepsilon(K) > 0$, depending only on K, the geometry (M, g) and H, such that if ξ_0 is a compatible H-structure on (M^n, g) satisfying

(i) $\|\nabla \xi_0\|_{L^\infty(M)} \leqslant K$ *and*

(ii) $\mathcal{D}(\xi_0) := \frac{1}{2}\|\nabla \xi_0\|^2_{L^2(M)} < \varepsilon(K)$,

then the harmonic H-flow with initial condition ξ_0 exists for all time $t \geqslant 0$ and converges smoothly to a torsion-free H-structure as $t \to \infty$.

Long-time existence under small initial torsion

Theorem (F-, Loubeau, Moreno, Sá Earp (2022))

Let (M^n, g) be a closed, oriented Riemannian n-manifold admitting a compatible H-structure ξ_0. Then, for every $\delta > 0$ there is $\varepsilon = \varepsilon(\delta, M^n, g, H) > 0$ such that if $\|\nabla\xi_0\|_{L^\infty(M)} < \varepsilon$ then the harmonic H-flow $\xi(t)$ starting from ξ_0 exists for all time $t \geqslant 0$ and converges smoothly to a harmonic H-structure ξ_∞ which furthermore satisfies $\|\nabla\xi_\infty\|_{L^\infty(M)} < \delta$.

Stability of torsion-free structures along the flow

Theorem (F-, Loubeau, Moreno, Sá Earp (2022))

Let (M^n, g) be a closed, oriented Riemannian n-manifold admitting a compatible H-structure. Then the following holds:

(i) *There is a constant $\kappa(M, g, H) > 0$ such that if ξ_0 is a compatible H-structure satisfying $\|\nabla\xi_0\|_{L^\infty(M)} < \kappa$ then the harmonic H-flow starting at ξ_0 exists for all $t \geqslant 0$ and converges smoothly to a torsion-free H-structure ξ_∞ as $t \to \infty$.*

(ii) *If (M^n, g) admits a compatible torsion-free H-structure $\overline{\xi}$, then for all $\delta > 0$ there is some $\overline{\varepsilon}(\delta, M, g, H) > 0$ such that for any compatible H-structure ξ_0 on (M^n, g) with $\|\xi_0 - \overline{\xi}\|_{C^1(M)} < \overline{\varepsilon}$ the harmonic H-flow with initial condition ξ_0 exists for all $t \geqslant 0$, satisfies the estimate $\|\xi_t - \overline{\xi}\|_{C^1(M)} < \delta$ for all $t \geqslant 0$, and converges smoothly to a torsion-free H-structure ξ_∞ as $t \to \infty$.*

Current projects and open questions

- $H = \mathrm{SU}(m) \subset \mathrm{SO}(2m)$ case (the $H = \mathrm{Sp}(k) \subset \mathrm{SO}(4k)$ case is being studied by Udhav Fowdar).
- The singularity structure of the harmonic H-flow: what are the types of singularities that can occur in each H-flow for different groups H? Produce examples of (shrinking or expanding) solitons.
- What are the topological obstructions (if any), according to the group H, for the existence of *divergence-free* H-structures?
- For each specific H, what is the role played by the initial condition in the behavior of the harmonic H-flow? Is there any special type of structures that are preserved along the harmonic H-flow? E.g., for $H = \mathrm{G}_2$, if φ_0 is a coclosed G_2-structure, is this property preserved along the harmonic G_2-flow with initial condition φ_0?
- Study the G_2 Laplacian co-flow with coclosed and $\mathrm{div}_g T = 0$ initial condition.