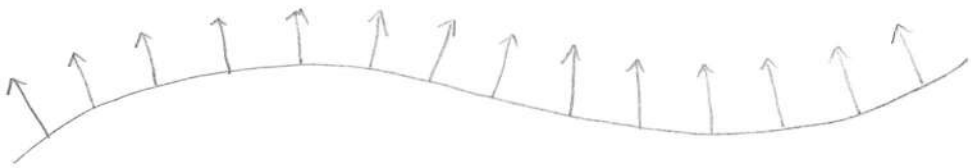
G_2-geometry via algebraic geometry 2

I/ Introduction

Let (M, g) be a Riem. mfd. (connected).

Let $\Omega(p)$ be the group of loops in M passing by $p \in M$.

$$\varphi : \Omega(p) \to O(T_p M)$$
$$\gamma \to \varphi_\gamma \to \text{parallel transport operator.}$$



Def: $Hol(M) := \mathrm{Im}\,\varphi$: the holonomy group of M.

Rk: Does not depend of p. Ex: $Hol(S^2) = SO(2)$.

Th (Berger, 1955): Let (M, g) be an irr. n-dim'l mfd with $\pi_1(M) = 0$. We assume that (M, g) is not symmetric.

Then: (i) $Hol(M) = SO(n)$

(ii) $Hol(M) = U(m)$ with $n = 2m$, $m \geq 2$

(iii) $Hol(M) = SU(m)$ with $n = 2m$, $m \geq 2$

(iv) $Hol(M) = Sp(m)$ with $n = 4m$, $m \geq 2$

(v) $\mathrm{Hol}(M) = Sp(m) \cdot Sp(1)$ with $n = 4m$, $m \geq 2$ Quaternion-Kähler

(vi) $\mathrm{Hol}(M) = G_2$ with $n = 7$

(vii) $\mathrm{Hol}(M) = \mathrm{Spin}(7)$ with $n = 8$.

Def: Let ϕ_0 be the 3-form on $\mathbb{R}^7$ (non-degenerate and positive) $x, y \overset{\tilde{g}}{\longmapsto} \frac{1}{6}(\omega \lrcorner x) \wedge (\omega \lrcorner y) \wedge \omega$.

$\tilde{g}$ sign-definite

$$\phi_0 = dx^{123} - dx^{145} - dx^{167} - dx^{246} + dx^{257} - dx^{347} - dx^{356}$$

$dx^{ijk} = dx^i \wedge dx^j \wedge dx^k$.

. $G_2 = \{ f \in O(7, \mathbb{R}) \mid f^* \phi_0 = \phi_0 \}$

. G_2-mfd : 7-dim'l mfd M such that $\mathrm{Hol}(M) \subseteq G_2$.

Ex: Let X be a Calabi-Yau 3-fold.

Then $X \times S^1$ is a G_2-manifold with $\mathrm{Hol}(X \times S^1) = SU(3) \subset G_2$,

indeed $\phi_0 = \underset{\text{Kähler form}}{dt \wedge \omega_X} + \underset{\text{holomorphic volume form}}{\mathrm{Re}(\Omega_X)}$

II Twisted connected sum construction of G_2-manifolds

1) Geometric construction

Prop (Corti, Haskins, Nordström, Pacini, 2013):

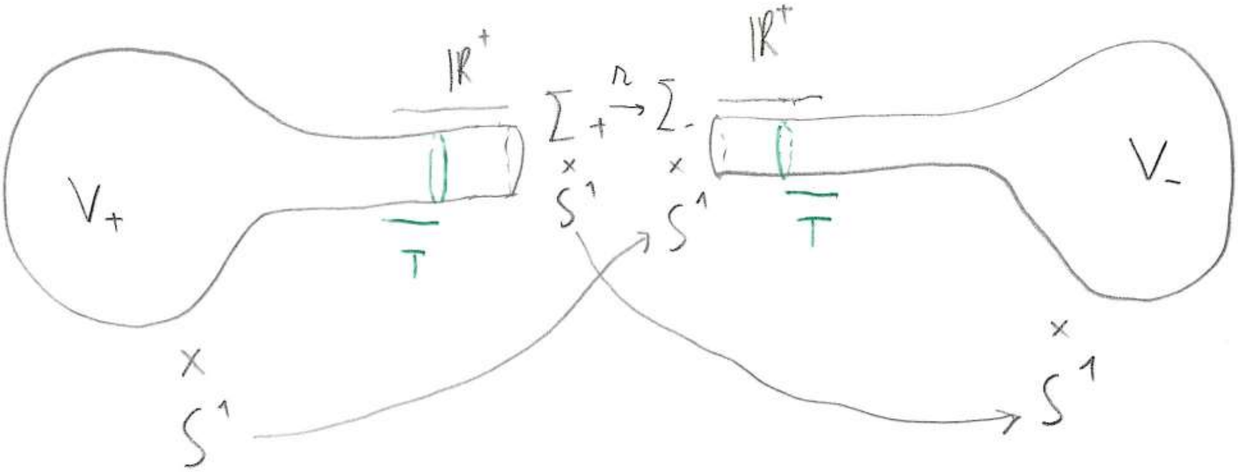
Let Y be a Fano 3-fold then $\dim|-K_Y| = g+2$ with $-K_3^3 = 2g-2$.

So let $Y_\pm$ be 2 Fano 3-folds and $|\Sigma_0^\pm, \Sigma_\infty^\pm| \subset |-K_{Y_\pm}|$ be a generic pencil with smooth base locus $\tau_\pm$.

- $Z_\pm \to Y_\pm$: blow-up in $\tau_\pm$ and $|\Sigma_0^\pm, \Sigma_\infty^\pm| \subset |-K_{Z_\pm}|$ stric-transform.

$\Rightarrow f_\pm : Z_\pm \to \mathbb{P}^1$.

- Let $x \in \mathbb{P}^1$ generic and $\Sigma^\pm = f_\pm^{-1}(x)$.
- $\Delta \ni x$ small disc, then $\Delta^* := \Delta \setminus \{x\} \simeq S^1 \times \mathbb{R}^+$
- And $f_\pm^{-1}(\Delta^*) \simeq S^1 \times \mathbb{R}^+ \times \Sigma^\pm$. $V_\pm = Z_\pm \setminus \Sigma_\pm$: calabi-Yau.



$r : \Sigma_+ \to \Sigma_-$: hyperkähler rotation.

<u>Th (Kovalev 2003):</u> $M := V_+ \times S^1 \cup_r V_- \times S^1$, then

$\mathrm{Hol}(M) = G_2$.

Idea of the proof:

On $\Sigma^{\pm}$, we have 3 possible Kähler structures $\omega_{\pm}^{I}$, $\omega_{\pm}^{J}$ and $\omega_{\pm}^{K}$.

Hyperkähler rotation $r^*(\omega_+^I) = \omega_-^J$; $r^*(\omega_+^J) = \omega_-^I$ and $r^*(\omega_+^K) = -\omega_-^K$

For $t \in]T-1, T[$:
$$F : \begin{array}{ccc} S^1 \times \mathbb{R}^+ \times S^1 \times \Sigma_+ & \longrightarrow & S^1 \times \mathbb{R}^+ \times S^1 \times \Sigma_- \\ (\theta, t, v, x) & \longrightarrow & (v, T-t, \theta, r(x)) \end{array}$$

$$\phi^{\pm} = d\theta \wedge dt \wedge dv + d\theta \wedge \omega_{\pm}^{I} + dv \wedge \omega_{\pm}^{J} + dt \wedge \omega_{\pm}^{K}.$$

We have $F^*(\phi_-) = \phi_+$

Q: How to find r?

2) Lattice criterion for the construction

Notation

- We consider $j_{\pm} : \Sigma_{\pm} \hookrightarrow Y_{\pm}$.
- $N_{\pm} := j_{\pm}^*(\mathrm{Pic}\, Y_{\pm}) \hookrightarrow H^2(\Sigma_{\pm}, \mathbb{Z}) \simeq U^3 \oplus E_8(-1)^2 := L_{K3}$ primitif.
- $\mathrm{Amp}_{\pm} \subset N_{\pm} \otimes \mathbb{R}$ the ample cone of $Y_{\pm}$.
- $N_0 \hookrightarrow N_{\pm}$ be a common primitive sub-lattice
- $W := \dfrac{N_+ \oplus N_-}{\{(x, -x) \mid x \in N_0\}}$: endowed with a $\mathbb{Q}$-valued bilinear form

given by $N_{\pm} \hookrightarrow W$ isometry.

Th (Corti, Haskins, Nordström, Pacini 2015):

- $N_0^{\perp} \cap N_{\pm} \cap Amp_{\pm} \neq \emptyset$
- W is a lattice
- $W \hookrightarrow L_{K3}$ primitive.

Then $\exists$ a hyperkähler rotation between 2 anti-canonical K3 surfaces $\Sigma_{\pm}$ of two deformations of $Y_{\pm}$.

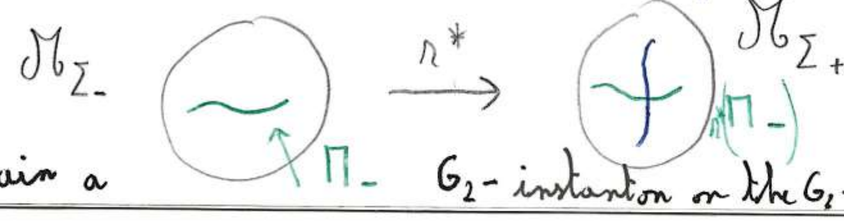
Idea: global Torelli th. for K3.

III G_2-instantons

Th (Sá Earp, Walpuski 2015): Let $F_{\pm} \to Z_{\pm}$ be a pair of bundles that verifies:

- Asymptotic stability: $F_{\pm}|_{\Sigma_{\pm}}$ is μ-stable.
- Compatibility: $c_1(F_{\pm}|_{\Sigma_{\pm}}) \in N_0$
- Inelasticity: $H^1(\mathcal{E}nd_0(F_{\pm})(-\Sigma_{\pm})) = 0$
- Transversality: $\mathcal{M}_{\Sigma_{\pm}}(v(F_{\pm}|_{\Sigma_{\pm}}))$: moduli space of stable bundle

$$\Pi_{\pm} := \{ G \in \mathcal{M}_{\Sigma_{\pm}}(v(F_{\pm}|_{\Sigma_{\pm}})) \mid \exists E \text{ bundle on } Z_{\pm},\ G = E|_{\Sigma_{\pm}} \}.$$



Π_+ et $r^*(\Pi_-)$ intersect transversaly.

Then, we obtain a G_2-instanton on the G_2-mfd associated to $\Sigma_{\pm}$.