Shopping for G_2-instantons on Aloff-Wallach spaces

Gonçalo Oliveira on joint work with Gavin Ball

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Pau, France, June 2023

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- $\varphi \in \Lambda^3_+ \rightsquigarrow$ a Riemannian metric g_φ (ex: $g_{\varphi_0} = \langle \cdot, \cdot \rangle$).
- M^7 a spin 7-manifold, a G_2-*structure* on M is a $\varphi \in \Omega^3(M)$ such that:

$$\forall p \in M, \quad \varphi_p \in \Lambda^3_+ T^*_p M.$$

From now on, let $\psi = *_\varphi \varphi \in \Omega^4(M)$.

Nearly parallel G_2-structures

- $\nabla\varphi$ determined by $d\varphi$ and $d\psi$ which are given by

$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *\tau_3, \quad d\psi = 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi,$$

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- However, for $\tau_0 \neq 0$, they need not be local minima of $\mathcal{E}$

$$\mathcal{E}(\mathcal{A}) = \int_M \langle F_\mathcal{A} \wedge F_\mathcal{A} \rangle \wedge \varphi + \|F_\mathcal{A} \wedge \psi\|^2_{L^2}.$$

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- Any complex line bundle L admits a unique G_2-instanton (the connection whose curvature is the harmonic representative of $c_1(L)$).

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Examples of nearly parallel φ?

Aloff-Wallach spaces

- Let $k, l \in \mathbb{Z}$ and $M_{k,l} = \mathrm{SU}(3)/\mathrm{U}(1)_{k,l}$, where

$$\begin{pmatrix} e^{ik\theta} & 0 & 0 \\ 0 & e^{il\theta} & 0 \\ 0 & 0 & e^{im\theta} \end{pmatrix}, \quad \text{and } k + l + m = 0.$$

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- Consider the left-invariant 1-forms $\{\omega_1, \dots, \omega_7, \xi\}$, with ξ the dual to the infinitesimal generator of $\mathrm{U}(1)_{k,l}$.
  The most general coclosed, homogeneous G_2-structure is

$$\varphi = ABC(\omega_{123} - \omega_{167} + \omega_{257} - \omega_{356}) - D\omega_4 \wedge (A^2\omega_{15} + B^2\omega_{26} + C^2\omega_{37}),$$

  for $(A, B, C, D) \in \mathcal{C} = (\mathbb{R}\backslash 0)^4/\mathbb{Z}_2^2$.

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for $(A, B, C, D) \in \mathcal{C} = (\mathbb{R}\backslash 0)^4/\mathbb{Z}_2^2$. Its associated the metric is
$$g_\varphi = A^2\left(\omega_1^2 + \omega_5^2\right) + B^2\left(\omega_2^2 + \omega_6^2\right) + C^2\left(\omega_3^2 + \omega_7^2\right) + D^2\omega_4^2.$$

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- φ nearly parallel $\Leftrightarrow$ quartic equation on $(A, B, C, D) \in \mathcal{C}$, with exactly 2 solutions for $k \neq \pm l$. These, correspond to strictly nearly parallel φ! (Cabrera, Monar and Swann)

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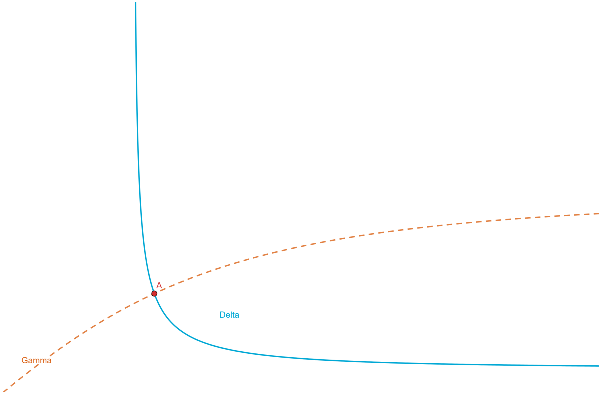

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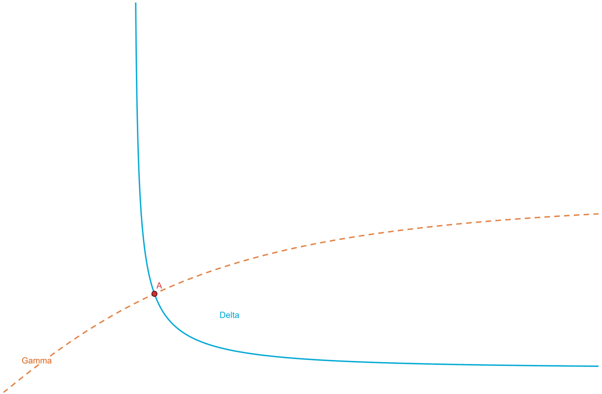

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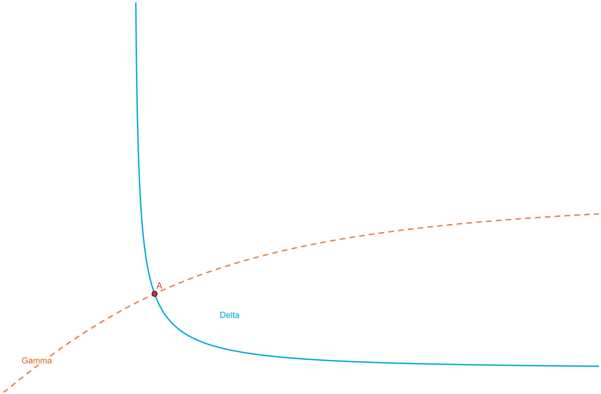

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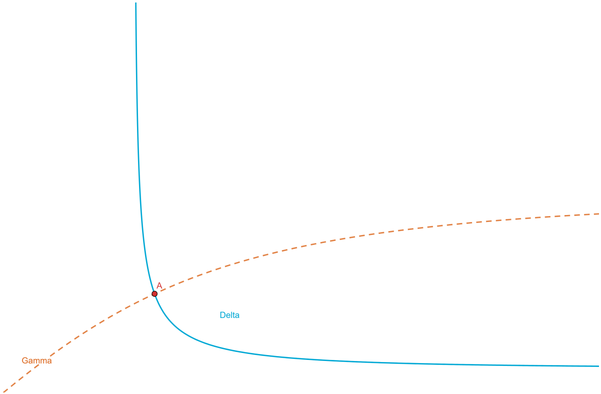

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Now we go to $G = SO(3)$!

Invariant $\mathrm{SO}(3)$ connections on $X_{k,l}$

- Homogeneous $\mathrm{SO}(3)$-bundles are parametrized by $\lambda_n : \mathrm{U}(1)_{k,l} \to \mathrm{SO}(3)$

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- Decompose into irreducible $\mathrm{U}(1)_{k,l}$-representations

$$\mathfrak{m} \cong \mathbb{R} \oplus \mathbb{C}_{k-l} \oplus \mathbb{C}_{l-m} \oplus \mathbb{C}_{m-k}, \qquad \mathfrak{so}(3) \cong \mathbb{R} \oplus \mathbb{C}_n,$$

  and Schur's lemma tells you the possible nonzero entries in Λ.

G_2-Instantons with $G = SO(3)$

For $k \neq l$, we can construct irreducible connections A if and only if n is either $k-l$, $l-m$, or $m-l$. In each of these cases there is a continuous function

$$\sigma_n : \mathcal{C} \to \mathbb{R},$$

with for example $\sigma_{k-l}(\varphi) = 3\left(\frac{m}{2} - \frac{\sqrt{6}}{\sqrt{k^2+l^2+m^2}}\frac{AD}{BC}\right)\Delta + \frac{k-l}{2}\Gamma$.

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Theorem (Classification)

Irreducible, invariant, G_2-instantons on P_n exist if and only if:

- *n is either $k - l$, $l - m$, or $m - k$, and*
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In this case, there are exactly 2 such instantons.

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Theorem (Deforming the G_2-structure)

Let $n = k - l$ and $\{\varphi(s)\}_{s\in\mathbb{R}} \subset \mathcal{C}$ a continuous family satisfying $\sigma_{k-l}(\varphi(s)) > 0$, for $s < 0$ and $\sigma_{k-l}(\varphi(s)) < 0$, for $s > 0$. Then, as $s \nearrow 0$, the two irreducible G_2-instantons on P_n merge into the same reducible and obstructed one.

Example: P_6 on $X_{1,-5}$

- $\varphi(A) \in \mathcal{C}$, the G_2-structures with $B = C = D = 1$.
- Existence of irreducible G_2-instantons is controlled by

$$\sigma_6(\varphi(A)) = (A^2 - 1)(12\sqrt{7}A - 42) > 0, \quad \text{i.e. for } A^2 < 1 \text{ or } A > \sqrt{7}/2.$$

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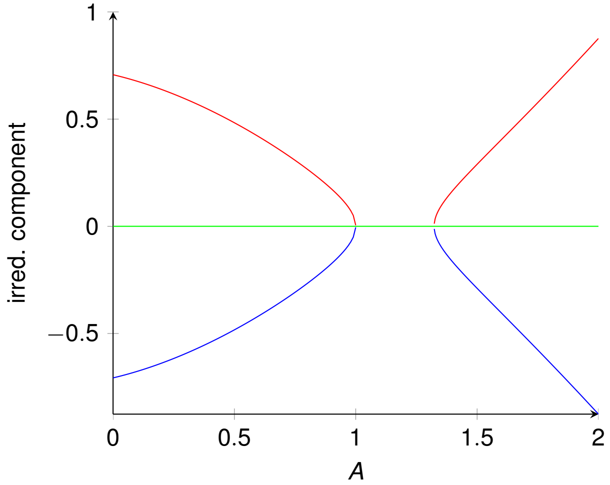

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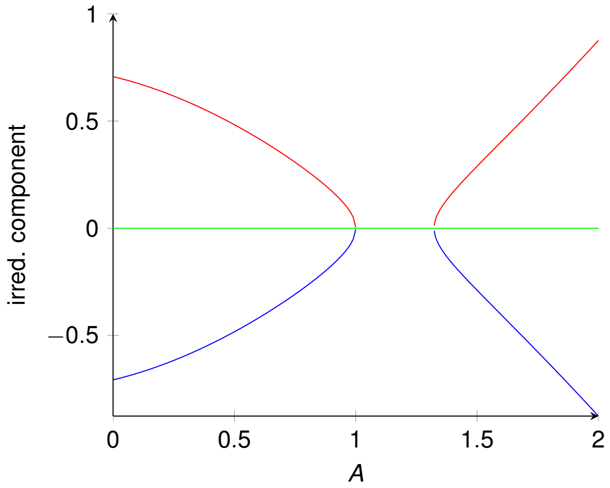

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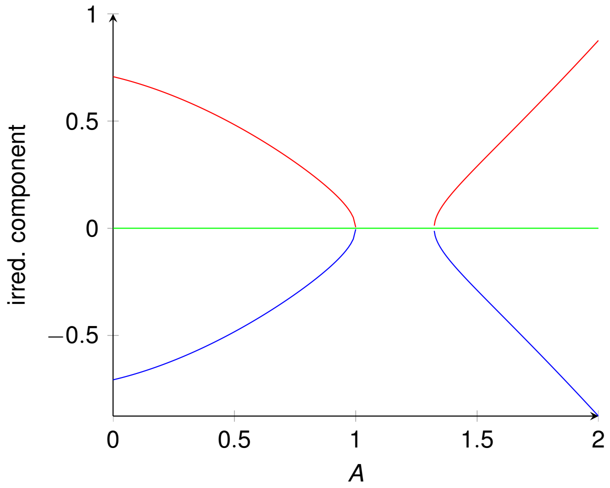

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$$A^{+} = 2.827, \ B^{+} = 2.197, \ C^{+} = 1.848, \ D^{+} = 2.668,$$

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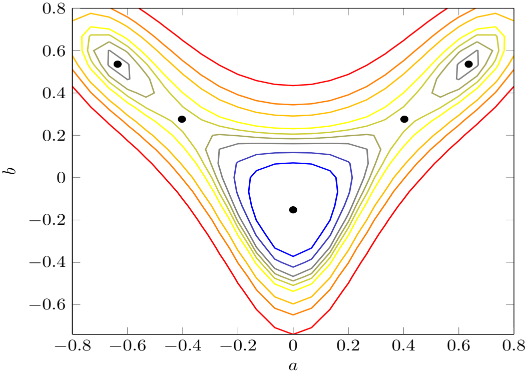
- This may be a general phenomena for the strictly nearly parallel G_2-structures on these $X_{k,l}$ (with $k \neq \pm l$). We did not try very hard to prove it!

Example: Non-minimal instantons for a nearly parallel φ on $X_{1,-1}$

Level sets of the invariant Yang-Mills functional on P_{-1}.



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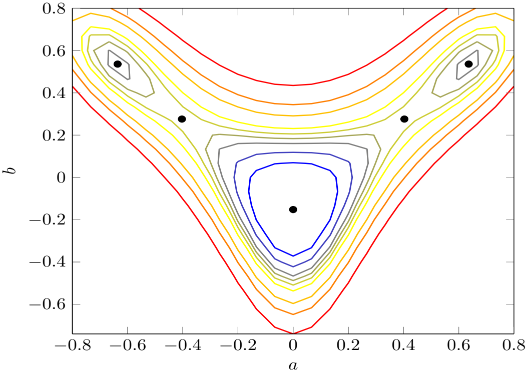

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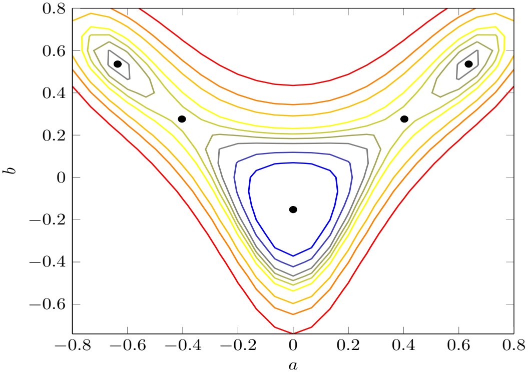

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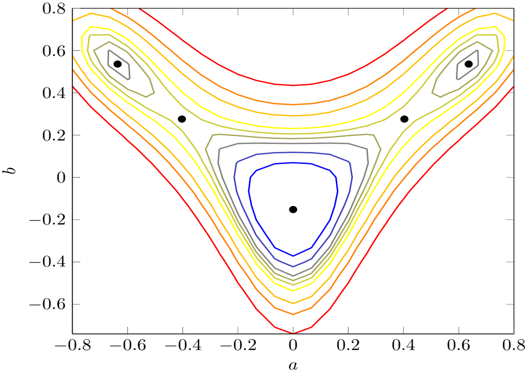

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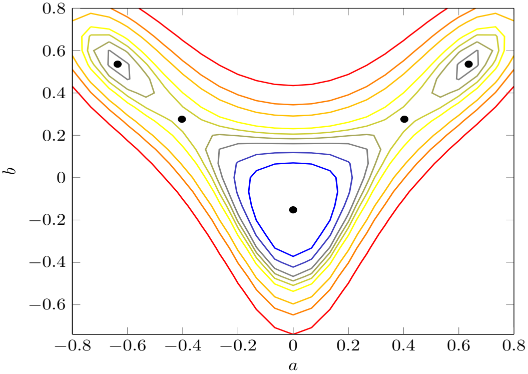

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Theorem (Distinguishing φ^{ts} and φ^{snp})

There are no irreducible invariant G_2-instantons with $G = SO(3)$ for φ^{ts}, but such G_2-instantons do exist for φ^{snp}.

Thank you!