UNIVERSIDADE FEDERAL DO CEARÁ

Laplacian coflow of G_2-structures on 7-manifolds

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G_2-structures

Definite 3-form

Let M be a Spin and orientable manifold. We define
$\Lambda^3_+ M := \bigsqcup_{x\in M} \Lambda^3_+(M)_x$, with
$\Lambda^3_+(M)_x := \{\varphi_x \in \Lambda^3 T^*_x M : \exists u \in \mathrm{Hom}(T_x M, \mathbb{R}^7), u^*\phi = \varphi_x\}$.

$$\phi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

G_2-structures

$\varphi \in \Omega^3_+(M) := \Gamma(\Lambda^3_+ M)$.

Associated metric

$\forall \varphi \in \Omega^3_+(M), \exists! g_\varphi$ such that $6 g_\varphi(X,Y)\mathrm{vol}_\varphi := (X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi$.

Dual 4-form

$\psi = *\varphi$

Decomposition of differential forms

- For 2-forms $\Omega^2(M) = \Omega^2_7(M) \oplus \Omega^2_{14}(M)$.

$$\Omega^2_7(M) := \{X \lrcorner \varphi : X \in \mathfrak{X}(M)\}$$
$$\Omega^2_{14}(M) := \{\omega \in \Omega^2(M) : \psi \wedge \omega = 0\}$$

- For 3-forms $\Omega^3(M) = \Omega^3_1(M) \oplus \Omega^3_7(M) \oplus \Omega^3_{27}(M)$,

$$\Omega^3_1(M) = \{f\varphi : f \in C^\infty(M)\}$$
$$\Omega^3_7(M) = \{*_\varphi(\alpha \wedge \varphi) : \alpha \in \Omega^1(M)\}$$
$$\Omega^3_{27}(M) = \{\alpha \in \Omega^3(M) : \alpha \wedge \varphi = 0 \quad \text{and} \quad \alpha \wedge *_\varphi = 0\} = i_\varphi(S^2_0(M)).$$

$$i_\varphi : S^2(T^*M) \to \Lambda^3(T^*M) \text{ such that } i_\varphi(h) = \frac{1}{2} h^l_i \varphi_{ljk} dx^i \wedge dx^j \wedge dx^k$$

Torsion forms

$$\mathrm{d}\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *\tau_3,$$
$$\mathrm{d}\psi = 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi.$$

where $\tau_0 \in \Omega^0(M)$, $\tau_1 \in \Omega^1(M)$, $\tau_2 \in \Omega^2_{14}(M)$ and $\tau_3 \in \Omega^3_{27}(M)$

Torsion classes of G_2-structures

- Torsion free: $\nabla^{g_\varphi}\varphi = 0 \Leftrightarrow d\varphi = 0, d*\varphi = 0.$
- Closed or calibrated: $d\varphi = 0$;
- Coclosed or cocalibrated: $d\psi = 0$;
- Nearly parallel: $d\varphi = c * \varphi$ for a constant c.

Full torsion tensor

2-tensor T_{lm} satisfying $\nabla_l \varphi_{abc} = T_{lm} g^{mn} \psi_{nabc}$.

$$T = \frac{\tau_0}{4} g_\varphi - *(\tau_1 \wedge \psi) - \frac{1}{2}\tau_2 - \frac{1}{4}\jmath(\tau_3), \tag{1}$$

Identities of the full torsion tensor

$\varphi \in \Omega^3_+(M)$ coclosed:

$$\begin{aligned} \mathrm{div} T =& \mathrm{d}(\mathrm{tr}\, T), \qquad \mathrm{Curl} T = (\mathrm{Curl} T)^t, \\ \mathrm{Ric} =& -\mathrm{Curl} T - T^2 + \mathrm{tr}(T) T, \quad R = (\mathrm{tr}\, T)^2 - |T|^2. \end{aligned} \tag{2}$$

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Laplacian coflow of G_2-structures

Laplacian coflow

Fixing an orientation given by φ_0 and $t \in [a, b)$

$$\frac{\partial}{\partial t}\psi_t = \Delta_t \psi_t = \mathrm{d}\mathrm{d}^{*_t}\psi_t + \mathrm{d}^{*_t}\mathrm{d}\psi_t.$$

If $d\psi_0 = 0$, then this property is preserved along the flow.

Volume functional

If M^7 is a compact manifold

$$V(\varphi) = \frac{1}{7}\int_M \varphi \wedge *\varphi$$

Laplacian coflow is the gradient flow of the volume functional. Then ψ defines a torsion-free G_2-structure if and only if it is a critical point of the functional V restricted to the cohomology class $[\psi] \in H^4(M, R)$.

Proposition-Grigorian

Under the flow (3), the evolution is given by

$$\frac{\partial g}{\partial t} = 2\mathrm{Curl}T + T\circ T + 2T^2 = -2\mathrm{Ric} + T\circ T + 2(\mathrm{tr}\,T)T,$$
$$\frac{\partial \mathrm{vol}}{\partial t} = \frac{1}{2}(|T|^2 + (\mathrm{tr}\,T)^2)\mathrm{vol},$$
$$\frac{\partial}{\partial t}T = \Delta T - 2\nabla(\mathrm{div}T) + Rm \circledast T + (\nabla T)\circledast T + T\circledast T\circledast T,$$

Modified Laplacian coflow

$$\frac{\partial \psi}{\partial t} = \Delta_\psi \psi + 2\mathrm{d}((A - \mathrm{tr}\,T)\varphi), \tag{3}$$

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Laplacian soliton for coclosed G_2-structure

(ψ, X, λ) satisfying

$$\Delta_\psi \psi = \mathcal{L}_X \psi + \lambda\psi = \mathrm{d}(X \lrcorner \psi) + \lambda\psi, \tag{4}$$

where $\mathrm{d}\psi = 0$, $\lambda \in \mathbb{R}$ and X is a vector field on M

It is natural to call a Laplacian soliton (ψ, X, λ) expanding if $\lambda > 0$; steady if $\lambda = 0$ and shrinking if $\lambda < 0$.

Proposition

If M^7 is compact, then there are no shrinking or steady soliton solutions, other than the trivial steady case of a torsion-free G_2-structure.

Theorem

Let φ be a coclosed G_2-structure on a compact manifold M and $X \in \mathfrak{X}(M)$. Then,

$$\mathcal{L}_X\psi = \frac{4}{7}(\mathrm{div}X)\psi \oplus (-\frac{1}{2}\mathrm{Curl}X + X\lrcorner T)^\flat \wedge \varphi \oplus * \mathrm{i}_\varphi\Big(\frac{1}{7}(\mathrm{div}X)g - \frac{1}{2}(\mathcal{L}_X g)\Big) \quad (5)$$

where $i_\varphi : S^2T^*M \to \Omega^3_1(M) \oplus \Omega^3_{27}(M)$ is the injective map.

Laplacian decomposition of coclosed G_2-structures

$$\Delta_\psi\psi = \frac{2}{7}((\mathrm{tr}\,T)^2 + |T|^2)\psi \oplus (\mathrm{d}\,\mathrm{tr}\,T) \wedge \varphi$$
$$\oplus *_\varphi \mathrm{i}_\varphi\Big(\mathrm{Ric} - \frac{1}{2}T \circ T - (\mathrm{tr}\,T)T + \frac{1}{14}\left((\mathrm{tr}\,T)^2 + |T|^2\right) g\Big)$$

Proposition

Let φ be a coclosed G_2-structure. If (ψ, X, λ) is a soliton of the Laplacian coflow then

$$\begin{aligned} \mathrm{div}T &= -\frac{1}{2}(\mathrm{Curl}X)^\flat + X \lrcorner T, \\ -\mathrm{Ric} + \frac{1}{2}T \circ T + (\mathrm{tr}\, T)T &= \frac{\lambda}{4}g + \frac{1}{2}\mathcal{L}_X g. \end{aligned} \tag{6}$$

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Sasakian manifolds

Contact form

A 1-form on $\mathbb{R}^{2n+1}$ that satisfies

$$\eta \wedge (\mathrm{d}\eta)^n \neq 0.$$

Contact manifold

A $2n+1$-dimensional manifold is a *contact manifold* if there exists a 1-form η, called a *contact* 1*-form*, on M such that

$$\eta \wedge (\mathrm{d}\eta)^n \neq 0$$

everywhere on M.

Almost contact structure $(M^{2n+1}, \eta, \xi, \Phi)$

It is a quadruple (M, η, ξ, Φ) where Φ is a tensor field of type $(1,1)$ (i.e, an endomorphism of TM), ξ is a vector field, and η is a 1-form which satisfies

$$\eta(\xi) = 1, \quad \Phi\xi = 0, \quad \eta \circ \Phi = 0 \tag{7}$$

$$\Phi \circ \Phi = -\mathrm{id} + \xi \otimes \eta. \tag{8}$$

A Riemannian metric on M is said to be *compatible* with the almost contact structure if for any fields X, Y on M we have

$$g(\Phi(X), \Phi(Y)) = g(X, Y) - \eta(X)\eta(Y). \tag{9}$$

A contact metric structure (M, η, ξ, Φ, g) satisfies

$$\omega(X, Y) = g(\Phi(X), Y) = \frac{1}{2}\mathrm{d}\eta(X, Y), \quad X, Y \in \mathcal{X}(M) \tag{10}$$

For a contact metric manifold (M, η, ξ, Φ, g) we take

$$\mathrm{vol}_g = \frac{\eta \wedge \omega^n}{n!} = \frac{1}{2^n n!} \eta \wedge (\mathrm{d}\eta)^n \tag{11}$$

as the Riemannian volume form.

K-contact

A contact metric structure (M, η, ξ, Φ, g) such that ξ is a killing vector field of g and we have

$$(\nabla_X \eta)(Y) = \frac{1}{2} \mathrm{d}\eta(X, Y) \tag{12}$$

$$r(X, \xi) = (2n)\eta(X),$$

$$g(R(X, \xi)Y, \xi) = g(X, Y) - \eta(X)\eta(Y), \tag{13}$$

where $X, Y \in \mathfrak{X}(M)$, ∇ is the covariant differentiation with respect g, r and R are the Ricci curvature tensor and Riemannian curvature tensor respectively.

A contact metric structure (ξ, η, Φ, g) is K-contact if and only if $\nabla\xi = -\Phi$.

Sasakian manifolds

The metric cone $(C(M), dr^2 + r^2 g, d(r^2\eta))$ is Kahler and it satisfies

$$(\nabla_X \Phi)Y = g(Y, \xi)X - g(X, Y)\xi, \tag{14}$$
$$R(X, \xi)Y = g(Y, \xi)X - g(X, Y)\xi \tag{15}$$

where $Y, Z \in \mathcal{X}(M)$.

A Sasakian manifold is necessarily a K-contact Riemanninan.

Contact Calabi-Yau manifolds (cCY)

$(M, \eta, \Phi, \Upsilon)$ such that

- (M^{2n+1}, η, Φ) is a Sasakian manifold.

$$g = \eta \otimes \eta + g_{\mathcal{D}} = \eta \otimes \eta + d\eta(\cdot, J\cdot) = \eta \otimes \eta + d\eta(\cdot, \Phi|_{\mathcal{D}}\cdot)$$

- Υ is a nowhere vanishing transversal $(n, 0)$-form on $\mathcal{D} = \ker \eta$:

$$\Upsilon \wedge \overline{\Upsilon} = c_n \omega^n, \quad d\Upsilon = 0,$$

where $c_n = (-1)^{\frac{n(n+1)}{2}} \mathrm{i}^n$ and $\omega = d\eta$.

Proposition

$(M^{2n+1}, \eta, \Phi, \Upsilon)$ be a cCY manifold. Then (M, η, ξ, Φ, g) is null-Sasakian and the metric g induced by (η, Φ) is a η-Einstein with $\lambda = 2$ and $\nu = 2n + 2$ and scalar curvature is equal to $2n - 1$.

Proposition

Let (M, η, Φ) be a compact simply-connected null-Sasakian η-Einstein manifold. Then $\mathrm{Hol}(\nabla) \subset \mathrm{SU}(n)$.

Proposition

A cCY manifold $(M^7, \eta, \Phi, \Upsilon)$ carries a cocalibrated G_2-structure

$$\varphi := \eta \wedge \omega + \mathrm{Re}\,\Upsilon,$$

with $\omega = d\eta$ and $d\varphi = \omega \wedge \omega$.

Its corresponding dual 4-form is given by

$$\psi = *\varphi = \frac{1}{2}\omega \wedge \omega - \eta \wedge \mathrm{Im}\,\Upsilon.$$

We want to consider the Laplacian coflow starting at the natural coclosed G_2-structure on a cCY

$$\varphi_0 = \varepsilon\eta_0 \wedge \omega_0 + \operatorname{Re}\Upsilon_0 \quad \text{and} \quad \psi_0 = \frac{1}{2}\omega_0^2 - \varepsilon\eta_0 \wedge \operatorname{Im}\Upsilon_0. \tag{16}$$

To this end, we consider the family of G_2-structures given by

$$\varphi_t = f_t h_t^2 \eta_0 \wedge \omega_0 + h_t^3 \operatorname{Re}\Upsilon_0, \tag{17}$$

for functions f_t, h_t depending only on time, with

$$f_0 = \varepsilon \quad \text{and} \quad h_0 = 1. \tag{18}$$

New solution

Theorem

Let $(M^7, \eta_0, \Phi_0, \Upsilon_0)$ be a cCY manifold. The family of coclosed G_2-structures φ_t on M^7 given by

$$\varphi_t = \varepsilon \mathrm{p}(t)^{-1} \eta_0 \wedge \omega_0 + \mathrm{p}(t)^3 \operatorname{Re} \Upsilon_0; \quad (19)$$

$$\psi_t = \frac{1}{2} \mathrm{p}(t)^4 \omega_0^2 - \varepsilon \eta_0 \wedge \operatorname{Im} \Upsilon_0; \quad (20)$$

where $\mathrm{p}(t) = 10t + 1$ and $t \in (-1/10, \infty)$, solves the Laplacian coflow with initial data determined by $\varphi_0 = \eta_0 \wedge \omega_0 + \operatorname{Re} \Upsilon_0$.

Associated metric and volume

In the setup, the Laplacian coflow on M^7, with initial data determined by φ_0, is solved by the following family of coclosed G_2-structures φ_t, with associated metric g_t, volume form vol_t and dual 4-form ψ_t:

$$g_t = \varepsilon^2 \mathrm{p}(t)^{-6} \eta_0^2 + \mathrm{p}(t)^2 g_{\mathcal{D}_0};$$
$$\mathrm{vol}_t = \varepsilon \mathrm{p}(t)^3 \eta_0 \wedge \mathrm{vol}_{\mathcal{D}_0},$$

where $\mathrm{p}(t) = (1 + 10\varepsilon^2 t)^{1/10}$ and $t \in (-\frac{1}{10\varepsilon^2}, \infty)$. Hence, the solution of the Laplacian coflow is immortal, with a finite time singularity (backwards in time) at $t = -\frac{1}{10\varepsilon^2}$.

Proposition

Let $\{\varphi_t\}$ be the Ansantz solution to the Laplacian coflow. Then

- Riemannian curvature is given by

$$|Rm_t|^2_{g_t} = (1 + 10\varepsilon^2 t)^{-2/5} |Rm_0^{\mathcal{D}_0}|^2_{g_0} + c_0 \varepsilon^4 (1 + 10\varepsilon^2 t)^{-2}$$

  for some constant $c_0 > 0$.
- if M is compact, then its volume is indeed strictly increasing in time, tending to infinity:

$$\mathrm{Vol}(M, g_t) \to \infty \quad \text{as } t \to \infty.$$

- Then the associated metric g_t is uniformly continuous (in t) on any compact interval contained in $(-\frac{1}{10\varepsilon^2}, \infty)$, but it is not uniformly continuous on $(-\frac{1}{10\varepsilon^2}, T)$ or (T, ∞) for any T.

Full torsion tensor

$$T_t = -\frac{3}{2}\varepsilon^3(1+10\varepsilon^2 t)^{-11/10}\eta_0^2 + \frac{1}{2}\varepsilon(1+10\varepsilon^2 t)^{-3/10} g_{\mathcal{D}_0}.$$

Proposition

Let $\{\varphi_t\}$ be the Ansatz solution. Then

$$\begin{aligned} |T_t|^2_{g_t} &= \frac{15}{4}\varepsilon^2(1+10\varepsilon^2 t)^{-1}, \\ |\nabla_t T_t|^2_{g_t} &= c_0\varepsilon^4(1+10\varepsilon^2 t)^{-2}, \\ \mathrm{div}_t T_t &= 0, \end{aligned}$$

where $c_0 > 0$ is a constant, ∇_t is the Levi-Civita connection of g_t and div_t is the divergence with respect to the metric g_t.

Chen's dilation for Ricci-Like flow

$$\Lambda(x,t) := \sup_M (|Rm(y,t)|^2_{g_t} + |T(y,t)|^4_{g_t} + |\nabla T(y,t)|^2_{g_t})^{\frac{1}{2}}$$

.

Proposition

suppose moreover M^7 is compact, and let $K := \sup_M |Rm_0^{\mathcal{D}_0}|_{g_0}$. Then there is a constant $c_0 > 0$, independent of ε, such that the quantity $\Lambda(t)$, along the Laplacian coflow solution is given by

$$\Lambda(t) = \left(K^2(1+10\varepsilon^2 t)^{-2/5} + c_0\varepsilon^4(1+10\varepsilon^2 t)^{-2}\right)^{1/2}.$$

Hence, the Laplacian coflow has a Type IIb infinite time singularity, unless $g_{\mathcal{D}_0}$ is flat, in which case it has a Type III infinite time singularity.

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Almost abelian

Let G be a Lie group, it is called *almost Abelian* if its Lie algebra $\mathfrak{g}$ admits an Abelian ideal $\mathfrak{h}$ of codimension 1.

we can consider that $e_7 \perp \mathfrak{h}$ and G_2-structure can be written as

$$\varphi = \omega \wedge e^7 + \rho^+ = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{245} - e^{236}, \quad (21)$$

where $\omega = e^{12} + e^{34} + e^{56}$ and $\rho_+ = e^{135} - e^{146} - e^{245} - e^{236}$ are the canonical $\mathrm{SU}(3)$–structure of $\mathfrak{h} \cong \mathbb{R}^6$

$$\psi := *\varphi = \frac{1}{2}\omega^2 + \rho_- \wedge e^7 = e^{1234} + e^{1256} + e^{3456} - e^{2467} + e^{2357} + e^{1457} + e^{1367}, \quad (22)$$

where $\rho_- = J^* \rho_+$ and J is the canonical complex structure on $\mathbb{R}^6$ defined by $\omega := \langle J \cdot, \cdot \rangle$

The transitive action of $\mathrm{GL}(\mathfrak{g})$ on the space of G_2-structures, defined by $h \cdot \varphi := (h^{-1})^* \varphi$ (for $h \in \mathrm{GL}(\mathfrak{g})$), yields an infinitesimal representation of the alternating 3-form

$$\Lambda^3(\mathfrak{g})^* = \theta(\mathfrak{gl}(\mathfrak{g}))\varphi \quad (23)$$

$\theta : \mathfrak{gl}(\mathfrak{g}) \to \mathrm{End}(\Lambda^3 \mathfrak{g}^*)$ is defined by

$$\theta(B)\varphi := \frac{d}{dt}\Big|_{t=0} e^{tB} \cdot \varphi = -\varphi(B\cdot, \cdot, \cdot) - \varphi(\cdot, B\cdot, \cdot) - \varphi(\cdot, \cdot, B\cdot). \quad (24)$$

Coclosed G_2-structures on almost Abelian Lie algebras are equivalent with the Lie bracket constrain $A \in \mathfrak{sp}(6, \mathbb{R})$ **[?]**, where

$$\mathfrak{sp}(\mathbb{R}^6) = \{A \in \mathfrak{gl}(\mathbb{R}^6) : \quad AJ + JA^t = 0 \quad \Leftrightarrow \quad \theta(A)\omega = 0\}.$$

Let $\mathcal{L} \simeq \mathfrak{gl}(\mathbb{R}^6)$ be the family of 7-dimensional almost Abelian Lie algebras. The subfamily $\mathcal{L}_{coclosed} \simeq \mathfrak{sp}(\mathbb{R}^6) \subset \mathcal{L}$ of coclosed G_2-structures is invariant under the bracket flow, which becomes equivalent to the following ODE for a one-parameter family of matrices $A = A(t) \in \mathfrak{sp}(\mathbb{R}^6)$:

$$\frac{d}{dt}A = -\left(\frac{1}{2}\operatorname{tr}(S_A)^2 + \frac{1}{4}(\operatorname{tr} JA)^2\right)A + \frac{1}{2}[A,[A,A^t]] + \frac{1}{2}[A, S_A \circ_6 S_A] \tag{25}$$

Consider the family of matrix

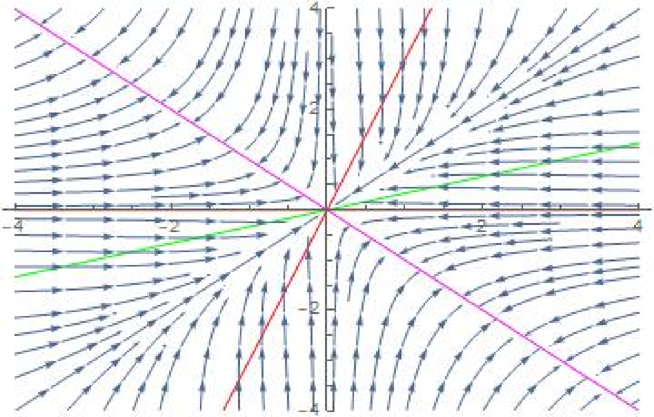
$$A = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & -B^t \end{array}\right] \quad \text{with} \quad B = \begin{bmatrix} 0 & x & 0 \\ y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad x, y \in \mathbb{R}$$

evolving under the bracket flow thus we obtain the nonlinear system given by

$$\dot{x} = -2x(3x-y)(x+y) \quad \text{and} \quad \dot{y} = 2y(x-3y)(x+y). \tag{26}$$

The resulting ODE is separable and the trajectories are level curves of

$$H(x(t), y(t)) = \frac{(y(t)-x(t))^2}{y(t)^3 x(t)^3} \tag{27}$$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