A Weitzenböck formula on Sasakian bundles

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Overview

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- $deg(E) = \frac{\mathbf{i}}{2\pi}\int_X \mathrm{Tr}(F_E) \wedge \omega^{n-1} \wedge \chi$, and $\mu(E) = \frac{deg(E)}{rank(E)}$, **stability**

Generalisation to $*(F_A) = \pm F_A$ for $n = 4$

($n > 4$) Higher dimensional Instantons

Choose $\sigma \in \Omega^{n-4}(M)$, A is σ-**instanton** if

$$* (\sigma \wedge F_A) = \lambda F_A, \quad \lambda \in \mathbb{R} \tag{1}$$

▸ Contact ▸ G2

($n > 4$) Foliated Instantons [2]

$(n-4)$ codimensional foliation $\mathcal{F}$ with characteristic form χ

$$* (\chi \wedge F_A) = \lambda F_A, \quad \lambda \in \mathbb{R} \tag{2}$$

Three natural notions of instanton

Contact instantons $n = 7$

In (1) ‣ InsEquation set $\sigma := \eta \wedge d\eta$

$$\Omega^2(M) = \Omega^2_1 \oplus \Omega^2_6 \oplus \Omega^2_8 \oplus \Omega^2_V.$$

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Theorem: $\mathcal{E} \to M$ Sasakian holomorphic on a cCY manifold; A Chern connection is tHYM $\iff$ it is a G_2-instanton. $\iff$ it is a SDCI.

Theorem (Theorem [1])

Let E be a G-bundle over a closed, connected Sasakian 7-manifold $(M, \mathcal{S})$, $\mathcal{M}^*$ the moduli space of irreducible SD contact instantons and $[A]$ a SD contact instanton, then:

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3. *If $H^2_B = 0$, $\mathcal{M}^*$ is smooth with* $\dim \mathcal{M}^* = -\,\mathrm{index}_T(A)$ ▸ *Vanishing*.

Deformation complex

We obtain an elliptic complex $(\mathrm{L}^\bullet, D)$

$$0 \to \mathrm{L}^0 \xrightarrow{D_0} \mathrm{L}^1 \xrightarrow{D_1} \mathrm{L}^2 \xrightarrow{D_2} \mathrm{L}^3 \to 0$$

which restricts to an transverse elliptic **basic complex**

$$0 \to \Omega^0_B(\mathfrak{g}_E) \xrightarrow{D_B} \Omega^1_B(\mathfrak{g}_E) \xrightarrow{D_B} (\Omega^2_{6\oplus 1})_B(\mathfrak{g}_E) \xrightarrow{D_B} 0$$

Remark

A Gysin sequence provides that $H^2 = 0$ implies $H^1 = 0$ for irreducible instantons, fortunately vanishing of the obstruction is obtained under the most reasonable condition $H^2_B = 0$

Definition

We define the **Laplacian** and the **transverse Laplacian** of D respectively by $\Delta := DD^* + D^*D$ and $\Delta_T := D_T D_T^* + D_T^* D_T - D_V^2$

Lemma

There exists an isomorphism $\phi\colon \mathcal{H}_T^{k,0} \xrightarrow{\sim} H_B^k$, where $\mathcal{H}_T^k = \ker(\Delta_T)$ and H_B^k is the cohomology of the basic complex.

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Local model [1]

Proposition

If $[A] \in \mathcal{M}^*$ is an irreducible SDCI such that the 'obstruction map' vanishes identically, then $\mathcal{M}^*$ is a smooth manifold near $[A]$.

Proposition

At an irreducible SD contact instanton such that the second basic cohomology group $H_B^2 = 0$, the obstruction Ψ vanishes.

Recall: If M is Sasakian $\mathcal{M}^*$ is Kähler

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Fix the following hypothesis: Let $(M^7, \mathcal{S})$ be a compact, connected, Sasakian manifold, $E \to M$ a Sasakian G-vector bundle, ∇ SDCI.

Proposition (Weitzenböck formula)

In coordinates, if $\varphi = \sum_{\gamma,\tau} \varphi_{\gamma\tau} dz^\gamma \wedge dz^\tau \in \Omega^{2,0}(\mathfrak{g}_E)$ *then*

$$(\Delta_{\partial_\nabla}\varphi)_{\mu\nu} = -\sum_{\alpha\beta} g^{\alpha\bar{\beta}} \widetilde{\nabla}_{\bar{\beta}} \widetilde{\nabla}_\alpha \varphi_{\mu\nu} - \mathcal{F}_{\mu\nu}(\varphi) - \mathcal{R}_{\mu\nu}(\varphi), \tag{3}$$

$\mathcal{F}, \mathcal{R} \in \mathrm{End}(\Omega^{2,0}(\mathfrak{g}_E))$ *depending on* F_∇ *and on the transverse Ricci curvature respectively, gives by*

$$\mathcal{F}(\varphi)_{\mu\nu} := \sum_{\alpha\beta} g^{\alpha\bar{\beta}} \left([\varphi_{\alpha\nu}, \mathrm{F}_{\mu\bar{\beta}}] - [\varphi_{\alpha\mu}, \mathrm{F}_{\nu\bar{\beta}}] \right)$$

and $\mathcal{R}(\varphi)_{\mu\nu} := \sum_{\alpha\beta} g^{\alpha\bar{\beta}} \left(\mathrm{R}_{\bar{\beta}\mu}\varphi_{\alpha\nu} - \mathrm{R}_{\bar{\beta}\nu}\varphi_{\alpha\mu} \right).$

Under the hypothesis to the Weitzenböck formula.

Theorem (Vanishing Theorem)

If the operator $\mathcal{F}$ and $\mathcal{R}$ are positive definite, then $\mathrm{H}_B^2 = 0$. Where H_B^2 is the basic cohomology of the basic ‣ Complex *associated to ∇*

Proposition

If $(M, \mathcal{S})$ is a compact, connected Ricci positive Sasakian manifold and E a $SU(n)$ Sasakian Vector bundle such that the irreducible SDCI $\nabla \in \mathcal{A}(E)$ induces a basic ‣ Complex *associated, then $\mathrm{H}_B^2 = 0$.*

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3-Sasakian manifold

$(4n+3)$ dimensional with 3 Sasakian structures ξ_1, ξ_2, ξ_3 (hyper-Kähler cone).

$$\begin{array}{ccc} M & \xrightarrow{\mathbb{S}^1} & \mathcal{Z} \\ {\scriptstyle \mathbb{RP}^3}\downarrow & \swarrow{\scriptstyle \mathbb{P}^1} & \\ X & & \end{array} \tag{4}$$

Geometry of the moduli space

Proposition

Let M be a compact, 3–Sasakian 7-manifold on smooth a smooth point A the 3–Sasakian structure induces the transverse quaternionic relations which endow $\mathcal{M}_a^*$ whit a hyper-Kähler structure.

Mercí!

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