University of Stuttgart
Institute of Geometry and Topology

Canonical connections and the Lichnerowicz Laplacian

Special geometries and gauge theories, Pau

23.06.2023

Paul Schwahn

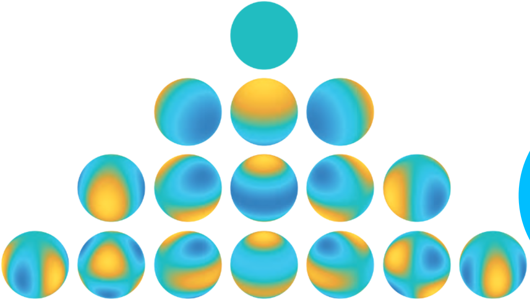

Image source: F. Pfaff, G. Kurz, U. D. Hanebeck,

Filtering on the Unit Sphere Using Spherical Harmonics

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mathematik

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(M^n, g) simply connected, non-symmetric such that $\mathrm{Hol}(\nabla^g) \curvearrowright TM$ irreducibly (holonomy irreducible). Then $\mathrm{Hol}(\nabla^g)$ is one of

$$\mathrm{SO}(n), \quad \mathrm{U}(n), \quad \mathrm{SU}(n), \quad \mathrm{Sp}(m)\,\mathrm{Sp}(1), \quad \mathrm{Sp}(m), \quad \mathrm{G}_2, \quad \mathrm{Spin}(7), \quad [\mathrm{Spin}(9)].$$

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By de Rham's theorem, a simply connected (M, g) with reducible holonomy is a Riemannian product of manifolds with irreducible holonomy.

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- If $\mathrm{Hol}(M, g) \subset G$, then ∇^g comes from a principal connection on a G-structure.

Nonintegrable geometries

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- $Z' \in \Omega^1(P, \mathfrak{g})$ is the natural metric connection on P.
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- A nonintegrable geometry is (M, g, P) such that $\Gamma \neq 0$.

Nonintegrable geometries: Characteristic connections

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Theorem (Friedrich–Ivanov 2002)

M admits a G-connection ∇^c with skew-symmetric torsion $T^c \in \Omega^3(M)$ if and only if $\Gamma \in \operatorname{im}\Theta$, where

$$\Theta : \ \Lambda^3\mathbb{R}^n \to \mathbb{R}^n \otimes \mathfrak{g}^\perp, \qquad \Theta(T) = \sum_i (e_i \lrcorner T) \otimes e_i,$$

(e_i) ONB of $\mathfrak{g}^\perp$. In this case

$$2\Gamma = -\Theta(T^c).$$

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- ∇^c has the same geodesics as ∇^g.

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- Integrable: $\mathrm{Hol}(\nabla^g) = \mathrm{SU}(n), \mathrm{Sp}(n), \mathrm{G}_2, \mathrm{Spin}(7)$ related to ∇^g-parallel spinors.

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  - E.g. Einstein–Sasaki manifolds, 3-Sasaki manifolds.

Nonintegrable geometries: parallel torsion

Theorem (Cleyton–Swann 2004)

(M, g, P) nonintegrable geometry with G-connection ∇ such that $\nabla T = 0$ and $\mathrm{Hol}(\nabla) \curvearrowright TM$ irreducibly. Then one of the following holds:

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In cases 1, 3, 4: $\nabla = \nabla^{\mathrm{c}}$*, hence* T *is skew-symmetric.*

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- Cone $\bar{M} = M \times_{r^2} \mathbb{R}_+$ has a parallel spinor, $\mathrm{Hol}(\nabla^{\bar{g}}) = \mathrm{G}_2$.

Nearly parallel G_2-manifolds

(Torsion-free) G_2-manifolds

- (M^7, g, φ) Riemannian manifold with compatible G_2-structure $\varphi \in \Omega^3_+(M)$ such that $d\varphi = 0$, $d^*\varphi = 0$.

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- $\exists$ Killing spinor, thus g is Einstein, $\mathrm{scal}_g = \frac{21}{8}\tau_0^2$.

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- ∇^c canonical G_2-connection, $\nabla^c g = 0$, $\nabla^c \varphi = 0$.
- $T^c = -\frac{\tau_0}{6}\varphi$.
- $\mathrm{Hol}(\nabla^c) \subset \mathrm{G}_2$, also weak holonomy G_2 in the sense of Gray.
- $\exists$ Killing spinor, thus g is Einstein, $\mathrm{scal}_g = \frac{21}{8}\tau_0^2$.

If (M, g, φ) is proper nearly parallel G_2 (not Sasakian):

- Cone $\bar{M} = M \times_{r^2} \mathbb{R}_+$ has a parallel spinor, $\mathrm{Hol}(\nabla^{\bar{g}}) = \mathrm{Spin}(7)$.

Reductive homogeneous spaces

G Lie group, H closed subgroup, $M = G/H$.

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- $\bar{T}_o(X, Y) = -[X, Y]_{\mathfrak{m}}, \quad X, Y \in \mathfrak{m} \cong T_o M$.

Naturally reductive homogeneous spaces

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- $\bar{T} = 0 \iff \nabla^c = \nabla^g \iff M$ symmetric.

Homogeneous classifications

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$$S^6 = \frac{\mathrm{G}_2}{\mathrm{SU}(3)}, \quad S^3 \times S^3 = \frac{\mathrm{SU}(2)^3}{\Delta\,\mathrm{SU}(2)}, \quad \mathbb{CP}^3 = \frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)\,\mathrm{U}(1)}, \quad F_{1,2} = \frac{\mathrm{SU}(3)}{T^2}.$$

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$$B = \frac{SO(5)}{SO(3)_{\mathrm{irr}}}, \qquad S^7_{\mathrm{sq}} = \frac{Sp(2) \times Sp(1)}{Sp(1) \times Sp(1)},$$

two non-isometric metrics on each of the Aloff–Wallach spaces

$$N_{k,l} = \frac{SU(3)}{S^1_{k,l}}, \qquad S^1_{k,l} = \{(z^k, z^l) \mid z \in S^1\} \subset T^2, \quad k, l \text{ coprime},$$

plus Einstein–Sasaki spaces (also known).

The Einstein–Hilbert action

M compact, oriented. For a Riemannian metric g on M, let

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- What is the type of these critical points?
- Consider the second variation S''_g. Also suppose $(M, g) \neq (S^n, g_{\text{round}})$.

The second variation of S

Suppose $(M, g) \ncong (S^n, g_{\text{round}})$.

- $T_g\mathscr{M}_1 = C_g^\infty(M)g \oplus L_{\mathfrak{X}(M)}g \oplus \mathscr{S}^2_{\mathrm{tt}}(M)$, where

$$C_g^\infty(M) := \{f \in C^\infty(M) \,|\, \textstyle\int_M f \,\mathrm{vol}_g = 0\},$$
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- The above splitting is orthogonal with respect to S_g''.
- $S_g'' > 0$ on $C_g^\infty(M)g$, i.e. g locally minimizes S along conformal change,
- $S_g'' = 0$ on $L_{\mathfrak{X}(M)}g$, i.e. S_g is unchanged under diffeomorphisms,
- S_g'' has finite coindex and nullity, i.e. is "‘mostly negative"’ in $\mathscr{S}^2_{\mathrm{tt}}(M)$.
- For $h \in \mathscr{S}^2_{\mathrm{tt}}(M)$, $S_g''(h,h) = -\frac{1}{2}\,(\Delta_{\mathrm{L}} h - 2Eh, h)_{L^2}$, where Δ_{L} is the Lichnerowicz Laplacian.

The Lichnerowicz Laplacian

(M, g) compact, oriented Riemannian manifold. ∇ Levi-Civita connection. R Riemannian curvature.

- Recall the Weitzenböck formula for the Hodge–de Rham Laplacian on differential forms $\Omega^p(M)$:

$$\Delta = d^*d + dd^* = \nabla^*\nabla + q(R).$$

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- (Standard) curvature endomorphism:

$$q(R) = \sum_i (\omega_i)_* \circ \widehat{R}(\omega_i)_*,$$

where $\widehat{R}: \Lambda^2TM \to \Lambda^2TM$ is the curvature operator of the first kind, and (ω_i) is a local ONB of $\mathfrak{so}(TM) \cong \Lambda^2TM$. Correspondence:

$$(X \wedge Y)_*Z = g(X, Z)Y - g(Y, Z)X.$$

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Both $\nabla^*\nabla$ and $q(R)$ are definable on any tensor bundle $\mathrm{Fr}(M) \times_{\mathrm{SO}(n)} V$, facilitating the definition

$$\Delta_{\mathrm{L}} := \nabla^*\nabla + q(R)$$

of the Lichnerowicz Laplacian.

Linear stability & infinitesimal deformability

For $h \in \mathscr{S}^2_{\mathrm{tt}}(M)$: $S''_g(h,h) = -\frac{1}{2}\left(\Delta_{\mathrm{L}} h - 2Eh, h\right)_{L^2}$.

- An Einstein metric g is called

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- **Question**: Does Δ_{L} have small eigenvalues on $\mathscr{S}^2_{\mathrm{tt}}(M)$?

Δ_{L} on Gray manifolds

(M^6, g, J) compact s.c. nK manifold. Normalize such that $\mathrm{scal}_g = 30$.

- Goal: Solve the differential system

$$\Delta_{\mathrm{L}} h = \lambda h, \qquad \delta_g h = 0 \tag{L}$$

in $h \in \mathscr{S}^2_0(M)$ for some $\lambda \leq 2E = 10$.

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$\rightsquigarrow$ System (L) is equivalent to

$$\begin{cases} (\bar\nabla^*\bar\nabla + q(\bar R))h^+ = (\lambda - 6)h^+ + (\delta\sigma)_{1,1} \circ J, \\ (\bar\nabla^*\bar\nabla + q(\bar R))h^- = (\lambda - 4)h^- - s, \\ \delta h^+ + \delta h^- = 0. \end{cases} \tag{L'}$$

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- Note: $\Delta^{\mathrm{c}} = \Delta$ on $\Omega^{1,1}_{0,\mathbb{R}} \cap \ker \delta_g$.

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Theorem (S. 2022)

Suppose $\lambda = 10 - \varepsilon$ *in system* (L). *The space of solutions of* (L) *is isomorphic to*

1. $E(\mu_1) \oplus E(\mu_2) \oplus E(\mu_3)$ *with*

$$\mu_{1,2} = 7 - \varepsilon \pm \sqrt{25 - 4\varepsilon}, \qquad \mu_3 = 6 - \varepsilon$$

   if $\varepsilon < \frac{25}{4}, \varepsilon \neq 6$,

2. $E(2) \oplus \ker \Delta\big|_{\Omega^3}$ *if* $\varepsilon = 6$,
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In particular $\varepsilon(g) \cong E(2) \oplus E(6) \oplus E(12)$ (Moroianu–Semmelmann 2011).

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⇝ Spectrum of $\bar{\Delta}$, hence of $\Delta = \Delta^{\mathrm{c}} = \bar{\Delta}$ on $\Omega^{1,1}_{0,\mathbb{R}} \cap \ker \delta_g$. Multiplicities given by $\dim \mathrm{Hom}_H(V_\gamma, V)$.

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The coindex of S''_g *on* $\mathscr{S}^2_{\mathrm{tt}}(M)$ *is*

- 2 *if* $M = S^3 \times S^3$ *(comes from harmonic* 3*-forms),*
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This proves that the coindex bounds by Semmelmann–Wang–Wang 2020 are sharp.

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Theorem (Wang–Wang 2018, Semmelmann–Wang–Wang 2020)

All compact s.c. homogeneous non-symmetric Einstein 7*-manifolds are unstable.*

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- How to make sense of $\mathcal{A}^*\bar{\nabla}$, $\mathcal{A}^*\mathcal{A}$?

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Theorem (S.-Semmelmann–Weingart 2022)

With the inclusion $\mathfrak{m}^{\otimes p} \subset \mathfrak{g}^{\otimes p}$,

$$\mathcal{A}^*\mathcal{A}\big|_{\mathfrak{m}^{\otimes p}} = \mathrm{pr}_{\mathfrak{m}^{\otimes p}} \mathrm{Cas}^{\mathfrak{g}}_{\mathfrak{g}^{\otimes p}}\big|_{\mathfrak{m}^{\otimes p}} - \mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{m}^{\otimes p}} - \mathrm{Der}_{\mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{m}}}.$$

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Theorem (S. 2023)

With the inclusion $C^\infty(G, \mathfrak{m}^{\otimes p})^H \subset C^\infty(G, \mathfrak{g}^{\otimes p}) \cong C^\infty(G) \otimes \mathfrak{g}^{\otimes p}$,

$$\mathcal{A}^*\bar{\nabla}\big|_{C^\infty(G,\mathfrak{m}^{\otimes p})^H} = \frac{1}{2}\mathrm{Cas}^{\mathfrak{g}}_{\ell} + \frac{1}{2}\mathrm{pr}_{\mathfrak{m}^{\otimes p}}\left(\mathrm{Cas}^{\mathfrak{g}}_{\mathfrak{g}^{\otimes p}} - \mathrm{Cas}^{\mathfrak{g}}_{C^\infty(G)\otimes\mathfrak{g}^{\otimes p}}\right) - \mathrm{Cas}^{\mathfrak{h}}_{\mathfrak{m}^{\otimes p}} .$$

Hence on symmetric tensors, if g is Einstein:

$$\Delta_{\mathrm{L}}\big|_{\mathrm{Sym}^p\mathfrak{m}} = \frac{3}{2}\mathrm{Cas}^{\mathfrak{g}}_{\ell} + \mathrm{pr}_{\mathrm{Sym}^p\mathfrak{m}}\left(\mathrm{Cas}^{\mathfrak{g}}_{\mathrm{Sym}^p\mathfrak{g}} - \frac{1}{2}\mathrm{Cas}^{\mathfrak{g}}_{C^\infty(G)\otimes\mathfrak{g}^{\otimes p}}\right) - \frac{3}{2}\mathrm{Cas}^{\mathfrak{h}}_{\mathrm{Sym}^p\mathfrak{m}} - pE + \frac{p}{4}.$$

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$\rightsquigarrow$ many new stability results for Einstein metrics with $\mathrm{scal}_g > 0$ (S. 2023).

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