Geometric flows of G_2-structures

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Workshop BRIDGES: Special geometries and gauge theories, University of Pau

June 19, 2023

based on a work in progress, joint work with

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&
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Introduction to G_2-structures

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Given local coordinates $x^1, \ldots, x^7$ on M, define

$$B_{ij} dx^1 \wedge \ldots \wedge dx^7 = \left(\frac{\partial}{\partial x^i} \lrcorner \varphi\right) \wedge \left(\frac{\partial}{\partial x^j} \lrcorner \varphi\right) \wedge \varphi$$

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φ is nondegenerate $\iff$ g_{ij} is a Riemannian metric.

G_2 structure $\leftrightsquigarrow$ "non-degenerate" 3-form $\varphi \rightsquigarrow g_\varphi$ and orientation nonlinearly.

Thus, we have a Hodge star operator $*_\varphi$ and dual 4-form $*_\varphi\varphi = \psi$.

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Let (M^7, φ) be a manifold with a G_2 structure φ and let ∇ be the Levi-Civita connection of g_φ. We call (M, φ) a G_2 manifold if $\nabla\varphi = 0$. $\nabla\varphi$ is interpreted as the torsion T of the G_2 structure.

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Aside: In 1955, Berger classified all possible Riemannian holonomies. The exceptional case G_2 was believed to not exist until seminal works by Bryant, Salamon, and Joyce, from 1987 to 1996.

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In particular,

$$\Omega^2 = \Omega^2_7 \oplus \Omega^2_{14}, \quad \Omega^3 = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}.$$

$$\begin{aligned} \Omega^2_7 = \{X \lrcorner \varphi \mid X \in \Gamma(TM)\} &= \{\beta \in \Omega^2 \mid *(\varphi \wedge \beta) = -2\beta\}, \\ \Omega^2_{14} = \{\beta \in \Omega^2 \mid *(\varphi \wedge \beta) = \beta\} &= \{\beta \in \Omega^2 \mid \beta \wedge \psi = 0\} \end{aligned}$$

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For $\sigma \in \Omega^k$ and $A = A_{ij} dx^i \otimes dx^j \in \mathcal{T}^2$, we define

$$(A \diamond \sigma)_{i_1 i_2 \cdots i_k} = A_{i_1}{}^p \sigma_{p i_2 \cdots i_k} + A_{i_2}{}^p \sigma_{i_1 p i_3 \cdots i_k} + \cdots + A_{i_k}{}^p \sigma_{i_1 i_2 \cdots i_{k-1} p},$$

in particular $(A \diamond \varphi)_{ijk} = A_i{}^p \varphi_{pjk} + A_j{}^p \varphi_{ipk} + A_k{}^p \varphi_{ijp}$.

Since $\mathcal{T}^2 \cong \Omega^0 \oplus S^2_0 \oplus \Omega^2_7 \oplus \Omega^2_{14}$, it can be proved that

$A \in \ker(\cdot \diamond \varphi) \iff A \in \Omega^2_{14}$

$A \mapsto A \diamond \varphi$ is an isomorphism between $S^2 \oplus \Omega^2_7$ and Ω^3

Introduction contd.

Thus, we can describe the 3-forms as

$$\Omega^3_1 = \{f\varphi \mid f \in \Omega^0\}, \quad \Omega^3_7 = \{A \diamond \varphi \mid A \in \Omega^2_7\} = \{X \lrcorner \psi \mid X \in \Gamma(TM)\},$$
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For $A \in \mathcal{T}^2$, we set

$$(\mathcal{V}A)_k = A_{ij}\varphi^{ij}{}_k.$$

Only the Ω^2_7 part of A contributes to $\mathcal{V}A$, and we call it the vector part of A. In fact,

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$$A_7 = \frac{1}{6}(\mathcal{V}A) \lrcorner \varphi.$$

Thus on (M, φ), *any* 3-form can be equivalently described by a pair (h, X) with h a symmetric 2-tensor and $X \in \Gamma(TM)$. We will write

$$\gamma = (h \diamond \varphi) + X \lrcorner \psi.$$

for a 3-form γ.

- The torsion T is a 2-tensor and is explicitly given as

$$T_{pq} = \frac{1}{24}\nabla_p \varphi_{ijk} {\psi_q}^{ijk} \qquad\qquad (\psi = *_\varphi \varphi)$$

Torsion of a G_2-structure

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- T satisfies a "Bianchi"-type identity

$$\nabla_i T_{jk} - \nabla_j T_{ik} = T_{ia} T_{jb} \varphi_a{}^{bk} + \frac{1}{2} R_{ijab} \varphi_k{}^{ab}.$$

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Theorem (Fernàndez–Gray)

(M, φ) is torsion-free, i.e., $T = 0$ ($\iff \nabla\varphi = 0$) if and only if $d\varphi = d\psi = 0$.

Suppose (M^7, φ) is a compact manifold with a G_2 structure.

Flows of G_2 structures

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A geometric flow is an evolution equation which improves a geometric structure, starting from a given one.

Given a G_2 structure (not necessarily torsion-free), it is natural to attempt to improve it in some sense to a "better" G_2 structure by using a geometric flow. (Compare: Ricci flow of metrics; mean curvature flow of immersions.)

- General study of flows of G_2 structures - Karigiannis, Q.J.M '09
  D.–Gianniotis–Karigiannis, '23

- Laplacian flow of closed G_2 structures - Bryant, '05, Bryant–Xu, '11
  ($\frac{\partial \varphi}{\partial t} = \Delta_\varphi \varphi, \quad d\varphi = 0$) Lotay–Wei, GAFA, CAG, JDG '15

- Laplacian co-flow of co-closed G_2 structures - Karigiannis–McKay–Tsui,
  ($\frac{\partial \psi}{\partial t} = -\Delta_\varphi \psi, \quad d\psi = 0$) DGA'12

- **Modified** Laplacian co-flow of co-closed G_2 structures - Grigorian,
  Adv.Math'13

- Isometric Flow of G_2-structures- D.–Gianniotis–Karigiannis, '19, independently by Grigorian, '19 $\rightsquigarrow$ using the theory of Octonionic bundles and by Loubeau–Sá Earp, '19 $\rightsquigarrow$ harmonic flow of geometric structures.

Recall that on (M, φ), any 3-form can be described by a pair (h, X), $h \in S^2(TM)$, $X \in \Gamma(TM)$. Thus, any flow of G_2-structures can be written as

$$\frac{\partial \varphi(t)}{\partial t} = (h(t) \diamond_t \varphi(t)) + X(t) \lrcorner \psi(t). \tag{GF}$$

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Facts: Along (GF), $\partial_t g(t) = 2h(t)$, $\partial_t g(t)^{-1} = -2h(t)$, $\partial_t \operatorname{vol}_t = \operatorname{tr} h(t) \operatorname{vol}_t$.

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One possible approach to write the most general (and reasonable) flow of G_2-structures is to classify all linearly independent second order differential invariants of a G_2-structure (upto lower order terms) and then take a linear combination of those which can be made into a 3-form.

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We use representation theoretic aspects of the Lie group G_2 to decompose the Riemann curvature tensor Rm and ∇T.

Definition

On (M, φ) $\exists$ another Ricci-type tensor F given explicitly as

$$F_{jk} = R_{abcd} \varphi^{ab}{}_{j} \varphi^{cd}{}_{k} \underbrace{=}_{symm. of\ \mathrm{Rm}} R_{cdab} \varphi^{ab}{}_{j} \varphi^{cd}{}_{k} = F_{kj}.$$

$\operatorname{tr}(F) = -2R$, $R =$scalar curvature. F has another geometric interpretation.

2nd order differential invariants of φ from Rm

The curvature decomposition is
$\mathrm{S}^2(\Lambda^2) = \mathrm{S}^2(\mathbf{7} \oplus \mathbf{14}) = \mathrm{S}^{\mathbf{2}}(\mathbf{7}) \oplus (\mathbf{7} \otimes \mathbf{14}) \oplus \mathrm{S}^{\mathbf{2}}(\mathbf{14})$ which can be further decomposed into irreducible G_2-representations as

$$(\mathbf{1} \oplus \mathbf{27}) \oplus (\mathbf{64} \oplus \mathbf{7} \oplus \mathbf{27}) \oplus (\mathbf{77} \oplus \mathbf{1} \oplus \mathbf{27}).$$

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However, it must be orthogonal to $\Lambda^4 = \mathbf{1} \oplus \mathbf{7} \oplus \mathbf{27}$ by the first Bianchi identity. This cuts down the curvature to an element of

$$\underbrace{\mathbf{1} \oplus \mathbf{27}}_{\text{Ricci}} \oplus \underbrace{\mathbf{27} \oplus \mathbf{64} \oplus \mathbf{77}}_{\text{Weyl}}.$$

That is, the Bianchi identity says that the **7** part is zero, that the two **1**'s are multiples of each other, and that the three **27**'s reduce to just two independent **27**'s. Only the **1** and the two **27** components can be made into a 3-form.

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Upshot: The only second order invariants from Rm which could appear for a flow of G_2-structures are: Rg, Ric_0 and W_{27}.

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Upshot: The only second order invariants from Rm which could appear for a flow of G_2-structures are: Rg, Ric_0 and W_{27}. Since

$$W_{27} = \frac{3}{92} F + \frac{3}{2 \cdot 115} Rg - \frac{3}{115} \mathrm{Ric}_0$$

we'll use Rg, Ric_0 and F.

2nd order differential invariants of φ from ∇T

- In a similar way we can decompose $\nabla T \in \Gamma(T^*M \otimes \mathcal{T}^2)$ into irreducible G_2-representations and look for those 2*nd* order differential invariants which can be made into a 3-form.

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• However, not all invariants obtained from Rm and ∇T are independent because these quantities are related by the G_2-Bianchi identity.

The G_2-Bianchi identity is

$$G_{qij} = \nabla_i T_{jq} - \nabla_j T_{iq} - T_{ia} T_{jb} \varphi^{ab}{}_q - \frac{1}{2} R_{ijab} \varphi^{ab}{}_q$$

G_{qij} are the components of a tensor $G \in \Gamma(T^*M \otimes \Lambda^2(T^*M))$, because G_{qij} is skew in i, j $\rightsquigarrow$ decomposed into two components $G^7 + G^{14}$, where $G^k \in \Gamma(T^*M \otimes \Lambda^2_k(T^*M))$ for $k = 7, 14$. Using the decompositions

$$\mathbf{7} \otimes \mathbf{7} = \mathbf{1} \oplus \mathbf{27} \oplus \mathbf{7} \oplus \mathbf{14} \qquad \text{and} \qquad \mathbf{7} \otimes \mathbf{14} = \mathbf{64} \oplus \mathbf{27} \oplus \mathbf{7},$$

we can therefore decompose $G = 0$ into seven independent relations.

Doing all these, we prove the following lemma.

All 2nd order differential invariants of φ

Lemma (D.–Gianniotis–Karigiannis, ’23)

Up to lower order terms, there are 6 independent 2nd order differential invariants which can be made into a 3-form. The choices are

$$h = Ric_0, Rg, F, \mathcal{L}_{\mathcal{V}T}g \qquad \textit{and} \qquad X = \operatorname{div} T, \operatorname{div} T^t.$$

Note that $\mathcal{V}T = \frac{1}{6}(T_7)\lrcorner\varphi$ *and* $(\operatorname{div} T)_k = \nabla^i T_{ik}$, $(\operatorname{div} T^t)_k = \nabla^i T_k{}^i$ *are vector fields on* M.

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These h's and X's appear in the first variation of the L^2-norm of the torsion components, i.e, in $\frac{d}{dt}\big|_{t=0} \int_M |T_i|^2 \operatorname{vol}$, $i = 1, 7, 14, 27$. The formulas are:

All 2nd order differential invariants of φ

Lemma (D.–Gianniotis–Karigiannis, '23)

Up to lower order terms, there are 6 independent 2nd order differential invariants which can be made into a 3-form. The choices are

$$h = Ric_0, Rg, F, \mathcal{L}_{\mathcal{V}T}g \qquad \text{and} \qquad X = \operatorname{div} T, \operatorname{div} T^t.$$

Note that $\mathcal{V}T = \frac{1}{6}(T_7)\lrcorner\varphi$ *and* $(\operatorname{div} T)_k = \nabla^i T_{ik},\ (\operatorname{div} T^t)_k = \nabla^i T_k{}^i$ *are vector fields on* M*.*

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$$\frac{d}{dt}\int_M |T_1|^2 \text{Vol} = \int_M h^{ip}((\operatorname{tr} T)^2 g_{ip} - 2 \operatorname{tr} T T_{ip}) \text{Vol}$$
$$- 2\int_M X^p(\operatorname{tr} T(\mathcal{V}T)_p + (\operatorname{div} T^t)_p) \text{Vol}$$

$$\frac{d}{dt}\int_M |T_7|^2 \,\mathrm{Vol} = \int_M 6h^{ip}\Big[(\mathcal{L}_{\mathcal{V}T}g)_{ip} + Rg_{ip} + \mathrm{tr}(T^2)g_{ip} - (\mathrm{tr}\,T)^2 g_{ip} + T_{la}T_{jq}\psi^{lajq}g_{ip}$$

$$- |\mathcal{V}T|^2 g_{ip} - 4(T_{\mathrm{skew}})_{is}T_p^{\ s} - 2T_{mn}T_{is}\psi^{mns}_{\ \ \ \ p}\Big]\,\mathrm{Vol}$$

$$+ \int_M 6X^q\Big[-2(\mathrm{div}\,T)_q + 2(\mathrm{div}\,T^t)_q + 2\nabla_p T_{mn}\psi^{pmn}_{\ \ \ \ q}$$

$$+ 4(T_{\mathrm{skew}})_{pq}\mathcal{V}T^p + 2(T^2)_{pn}\varphi^{pn}_{\ \ \ q}\Big]\,\mathrm{Vol}$$

$$\frac{d}{dt}\int_M |T_{14}|^2\,\mathrm{Vol} = \int_M h^{ip}\Big[R_{ip} - \frac{11}{2}(\mathcal{L}_{\mathcal{V}T}g)_{ip} - \frac{1}{4}F_{ip} - 6Rg_{ip} + 2(T^2)_{pi} - (T\circ T^t)_{ip}$$

$$- \mathrm{tr}\,TT_{pi} - \frac{1}{2}T_{ms}T_{nt}\varphi^{mn}_{\ \ \ i}\varphi^{st}_{\ \ \ p} - 2T_{km}(T_{\mathrm{skew}})_{pq}\psi^{kmq}_{\ \ \ \ i}$$

$$+ 12T_{mn}T_{is}\psi^{mns}_{\ \ \ \ p} + 24(T_{\mathrm{skew}})_{is}T_p^{\ s} + \frac{1}{2}|T|^2 g_{ip} - \frac{13}{2}\mathrm{tr}(T^2)g_{ip}$$

$$+ 6(\mathrm{tr}\,T)^2 g_{ip} + 6|\mathcal{V}T|^2 g_{ip} - 6T_{la}T_{jq}\psi^{lpjq}g_{ip}\Big]\,\mathrm{Vol}$$

$$- X^q\int_M \Big[13(\mathrm{div}\,T)_q - 13(\mathrm{div}\,T^t)_q - 24(T_{\mathrm{skew}})_{pq}(\mathcal{V}T)^p$$

$$- 13(T^2)_{pl}\varphi^{pl}_{\ \ \ q} - 12\nabla_p T_{mn}\psi^{pmn}_{\ \ \ \ q}\Big]\,\mathrm{Vol}$$

$$
\begin{aligned}
\frac{d}{dt}\int_M |T_{27}|^2 \,\mathrm{Vol} = \int_M h^{ip}\Big[& R_{ip} + \frac{1}{2}(\mathcal{L}_{\mathcal{V}T}\, g)_{ip} + \frac{1}{4}F_{ip} + \frac{1}{2}T_{ms}T_{nt}\varphi^{mn}{}_{i}\varphi^{st}{}_{p} - \frac{5}{7}\,\mathrm{tr}\, T T_{pi} \\
& - 2T_{km}(T_{\mathrm{sym}})_{pq}\psi^{kmq}{}_{i} - (T\circ T^t)_{ip} + \frac{1}{2}|T|^2 g_{ip} + \frac{1}{2}\mathrm{tr}(T^2) g_{ip} \\
& - \frac{1}{7}(\mathrm{tr}\, T)^2 g_{ip}\Big]\,\mathrm{Vol} \\
& - \int_M X^q\Big[(\mathrm{div}\, T)_q + \frac{9}{7}(\mathrm{div}\, T^t)_q + T^2_{pl}\varphi^{pl}{}_{q} + \frac{2}{7}\,\mathrm{tr}\, T(\mathcal{V}T)_q \\
& + \frac{2}{7}(\mathcal{V}T)^p T_{qp}\Big]\,\mathrm{Vol}\,.
\end{aligned}
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$$\frac{d}{dt}\int_M |T_{27}|^2 \,\mathrm{Vol} = \int_M h^{ip}\Big[R_{ip} + \frac{1}{2}(\mathcal{L}_{\mathcal{V}T}g)_{ip} + \frac{1}{4}F_{ip} + \frac{1}{2}T_{ms}T_{nt}\varphi^{mn}{}_i\varphi^{st}{}_p - \frac{5}{7}\operatorname{tr} T T_{pi}$$

$$- 2T_{km}(T_{\mathrm{sym}})_{pq}\psi^{kmq}{}_i - (T \circ T^t)_{ip} + \frac{1}{2}|T|^2 g_{ip} + \frac{1}{2}\operatorname{tr}(T^2)g_{ip}$$

$$- \frac{1}{7}(\operatorname{tr} T)^2 g_{ip}\Big]\,\mathrm{Vol}$$

$$- \int_M X^q\Big[(\operatorname{div} T)_q + \frac{9}{7}(\operatorname{div} T^t)_q + T^2_{pl}\varphi^{pl}{}_q + \frac{2}{7}\operatorname{tr} T(\mathcal{V}T)_q$$

$$+ \frac{2}{7}(\mathcal{V}T)^p T_{qp}\Big]\,\mathrm{Vol}\,.$$

Thus, we are led to define the following family of flows of G_2-structures.

[Flows of G_2-structures]

Let (M^7, φ_0) be a compact manifold. The general flow of G_2-structures is the initial value problem

$$\frac{\partial \varphi}{\partial t} = (-\operatorname{Ric} + a\mathcal{L}_{\mathcal{V}T}g + \beta F) \diamond \varphi + (b_1 \operatorname{div} T + b_2 \operatorname{div} T^t) \lrcorner \psi + \text{l.o.t.} \tag{GGF}$$

$$\varphi(0) = \varphi_0$$

with $a, \beta, b_1, b_2 \in \mathbb{R}$.

Remark: We do not put any condition on φ (like $d\varphi = 0, d * \varphi = 0$ or isometric).

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Remark: We do not put any condition on φ (like $d\varphi = 0, d * \varphi = 0$ or isometric).

Special cases

- $a = \beta = b_1 = b_2 = 0$ gives the usual **Ricci flow of G_2-structures.** The analytic properties of this flow is well-understood, in particular, we have short-time existence and uniqueness, *a priori* estimates and a compactness theorem for solutions.

Special cases contd.

$$\frac{\partial \varphi}{\partial t} = (-\operatorname{Ric} + a\mathcal{L}_{\mathcal{V}T} g + \beta F) \diamond \varphi + (b_1 \operatorname{div} T + b_2 \operatorname{div} T^t) \lrcorner \psi + \text{l.o.t.}$$

- $a = \beta = b_2 = 0$, $b_1 = 1$ and no Ric term gives the **isometric/harmonic flow of G_2-structures** $\rightsquigarrow$ **negative gradient flow of $\varphi \mapsto \int_M |T|^2 \operatorname{Vol}$ restricted to $[[\varphi_0]]_{\text{iso}}$.** Analytic properties well-understood and we have a monotonicity formula and entropy functional.

Special cases contd.

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- $a = \beta = b_2 = 0$ gives the **Ricci flow coupled with the isometric flow of G_2-structures.** We prove short-time existence and uniqueness of solutions and *a priori* estimates.

Special cases contd.

$\frac{\partial \varphi}{\partial t} = (-\operatorname{Ric} + a\mathcal{L}_{\mathcal{V}T}g + \beta F) \diamond \varphi + (b_1 \operatorname{div} T + b_2 \operatorname{div} T^t) \lrcorner \psi + \text{l.o.t.}$

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- $a = -\frac{1}{2}$, $\beta = 0$, $b_1 = 1$, $b_2 = 0$ so we have $\partial_t \varphi = (-\operatorname{Ric} - \frac{1}{2}\mathcal{L}_{\mathcal{V}T}g) \diamond \varphi + \operatorname{div} T \lrcorner \psi + \text{l.o.t.}$ $\rightsquigarrow$ **negative gradient flow of** $\varphi \mapsto \int_M |T|^2 \operatorname{Vol}$ **on** ***all*** G_2**-structures.** Studied by Weiss–Witt (2012). We have short-time existence and uniqueness of solutions.

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What can we say about the short-time existence and uniqueness of solutions of (GGF) in general?

Theorem (D.–Gianniotis–Karigiannis, '23)

Let (M, φ_0) be a compact 7-manifold with a G_2-structure φ_0. Then there exists a unique $\varphi(t)$, $t \in [0, \varepsilon)$, such that

$$\frac{\partial \varphi(t)}{\partial t} = (-\,Ric + a\mathcal{L}_{\mathcal{V}T} g + \beta F) \diamond \varphi + (b_1 \operatorname{div} T + b_2 \operatorname{div} T^t) \lrcorner \psi$$

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provided that $0 \leq b_1 - a - 1 < 4$, $b_1 + b_2 \geq 1$ and $|\beta| < \dfrac{c}{4}$, where $c = 1 - \frac{1}{4}(b_1 - a - 1)$.

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Idea of the proof:

• Let the RHS of (GGF) be P_φ. Calculate the principal symbols of the operators involved: $\mathrm{Ric}, \mathcal{L}_{\mathcal{V}T}g, F, \operatorname{div} T, \operatorname{div} T^t$. It turns out that $\dim \ker\ (\sigma(DP_\varphi)(h, X)) \geq 7$ because of diffeomorphism invariance of the tensors involved.

• We prove that dim ker $(\sigma(DP_{\varphi})(h, X)) = 7$ and hence the failure of parabolicity of (GGF) is only due to diffeomorphism invariance of the tensors involved. Remark: We needed to introduce a new operator to show this; we have $B_1 : S^2(T_x^* M) \to T_x^* M$ with $B_1(h)_k = \xi_a h^a{}_k - \frac{1}{2}\xi_k \operatorname{tr} h$ which is the usual *Bianchi operator* in the Ricci-flow. We introduce $B_2 : T_x^* M \to T_x^* M$ by $B_2(X)_k = \xi_a X_b \varphi^{ab}{}_k$ and use both of these.

Proof contd.

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• Because of above, we can use the DeTurck's trick: look at the modified operator $P_\varphi + \mathcal{L}_W\varphi$ with $W \in \Gamma(TM)$ given by

$$W^k = g^{ij}\left(\Gamma_{ij}^k - \bar{\Gamma}_{ij}^k\right) - 2a\mathcal{V}T^k$$

where $\bar{\Gamma}$ are the Christoffel symbols w.r.t. a fixed background G_2-structure, e.g., φ_0.

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• The symbol of $P_\varphi + \mathcal{L}_W \varphi$ is a multiple of Id and hence we can prove short-time existence and uniqueness using DeTurck's trick.

Future Problems

• One can ask the same questions for general flows of $\mathrm{Spin}(7)$-structures on 8-manifolds. This has been done by D. '23 and is an upcoming work.

• Find the flow with the "nicest" evolution of the torsion. This will involve the lower order terms as well. We have a "heat-type" equation for T along the negative gradient flow of $\varphi \mapsto \int_M |T|^2 \mathrm{Vol}$ functional ($\beta = b_2 = 0, a = -\frac{1}{2}, b_1 = 1$ case).

• A monotone quantity, just like the case of the isometric flow and if possible, an entropy functional, "smallness" of which guarantees long time existence.

• Examples of solutions and solitons.

• Dynamical stability of torsion-free G_2-structures along the flows considered here (already done for the neg. grad. flow by Weiss–Witt (2012)).

• Applications of the flow to 3, 4 and 6 dimensions à la Fine–Yao, Lambert–Lotay and Picard–Suan.

Thank you for your attention.