Non-associative gauge theory

Sergey Grigorian

University of Texas Rio Grande Valley, Edinburg, TX, USA

BRIDGES Workshop, Pau, June 19, 2023

Motivation

- Gauge theory is essentially the study of connections on principal G-bundles over manifolds, for a Lie group G.

Motivation

- Gauge theory is essentially the study of connections on principal G-bundles over manifolds, for a Lie group G.
- Gauge transformations of a connection are given by the action of the group of sections of an associated bundle.

Motivation

- Gauge theory is essentially the study of connections on principal G-bundles over manifolds, for a Lie group G.
- Gauge transformations of a connection are given by the action of the group of sections of an associated bundle.
- Here we build up a gauge theory that admits *non-associative* transformations.

Motivation

- Gauge theory is essentially the study of connections on principal G-bundles over manifolds, for a Lie group G.
- Gauge transformations of a connection are given by the action of the group of sections of an associated bundle.
- Here we build up a gauge theory that admits *non-associative* transformations.
- Algebraically, this is based on the theory of *loop*, which are non-associative analogues of groups.

- An example of such a theory first arose in the study of 7-dimensional manifolds with a G_2-structure.

- An example of such a theory first arose in the study of 7-dimensional manifolds with a G_2-structure.
- In particular, each G_2-structure defines a unique Riemannian metric.

- An example of such a theory first arose in the study of 7-dimensional manifolds with a G_2-structure.
- In particular, each G_2-structure defines a unique Riemannian metric.
- However, the set of G_2-structures that are compatible with a fixed metric can be parametrized by sections of a *unit octonion bundle*.

- An example of such a theory first arose in the study of 7-dimensional manifolds with a G_2-structure.
- In particular, each G_2-structure defines a unique Riemannian metric.
- However, the set of G_2-structures that are compatible with a fixed metric can be parametrized by sections of a *unit octonion bundle*.
- The relationship between the group G_2 and the octonions is of course natural - G_2 is the automorphism group of the non-associative octonion algebra.

- An example of such a theory first arose in the study of 7-dimensional manifolds with a G_2-structure.
- In particular, each G_2-structure defines a unique Riemannian metric.
- However, the set of G_2-structures that are compatible with a fixed metric can be parametrized by sections of a *unit octonion bundle*.
- The relationship between the group G_2 and the octonions is of course natural - G_2 is the automorphism group of the non-associative octonion algebra.
- It has been noted [SG'17] that the choice of the G_2-structure in a fixed metric class can be regarded as choice of gauge, with gauge transformations given by octonion multiplication.

- An example of such a theory first arose in the study of 7-dimensional manifolds with a G_2-structure.
- In particular, each G_2-structure defines a unique Riemannian metric.
- However, the set of G_2-structures that are compatible with a fixed metric can be parametrized by sections of a *unit octonion bundle*.
- The relationship between the group G_2 and the octonions is of course natural - G_2 is the automorphism group of the non-associative octonion algebra.
- It has been noted [SG'17] that the choice of the G_2-structure in a fixed metric class can be regarded as choice of gauge, with gauge transformations given by octonion multiplication.
- This talk is based on the following papers:

- An example of such a theory first arose in the study of 7-dimensional manifolds with a G_2-structure.
- In particular, each G_2-structure defines a unique Riemannian metric.
- However, the set of G_2-structures that are compatible with a fixed metric can be parametrized by sections of a *unit octonion bundle*.
- The relationship between the group G_2 and the octonions is of course natural - G_2 is the automorphism group of the non-associative octonion algebra.
- It has been noted [SG'17] that the choice of the G_2-structure in a fixed metric class can be regarded as choice of gauge, with gauge transformations given by octonion multiplication.
- This talk is based on the following papers:
  1. SG, *G_2-structures and octonion bundles*, Adv. Math, 2017, arXiv:1510.04226

- An example of such a theory first arose in the study of 7-dimensional manifolds with a G_2-structure.
- In particular, each G_2-structure defines a unique Riemannian metric.
- However, the set of G_2-structures that are compatible with a fixed metric can be parametrized by sections of a *unit octonion bundle*.
- The relationship between the group G_2 and the octonions is of course natural - G_2 is the automorphism group of the non-associative octonion algebra.
- It has been noted [SG'17] that the choice of the G_2-structure in a fixed metric class can be regarded as choice of gauge, with gauge transformations given by octonion multiplication.
- This talk is based on the following papers:
  1. SG, *G_2-structures and octonion bundles*, Adv. Math, 2017, arXiv:1510.04226
  2. SG, *Smooth Loops and Smooth Bundles,* Adv. Math., 2021, arXiv: 2008.08120

- An example of such a theory first arose in the study of 7-dimensional manifolds with a G_2-structure.
- In particular, each G_2-structure defines a unique Riemannian metric.
- However, the set of G_2-structures that are compatible with a fixed metric can be parametrized by sections of a *unit octonion bundle*.
- The relationship between the group G_2 and the octonions is of course natural - G_2 is the automorphism group of the non-associative octonion algebra.
- It has been noted [SG'17] that the choice of the G_2-structure in a fixed metric class can be regarded as choice of gauge, with gauge transformations given by octonion multiplication.
- This talk is based on the following papers:
  1. SG, *G_2-structures and octonion bundles*, Adv. Math, 2017, arXiv:1510.04226
  2. SG, *Smooth Loops and Smooth Bundles,* Adv. Math., 2021, arXiv: 2008.08120
  3. SG, *The Coulomb gauge in non-associative gauge theory,* arXiv:2303.00664

- An example of such a theory first arose in the study of 7-dimensional manifolds with a G_2-structure.
- In particular, each G_2-structure defines a unique Riemannian metric.
- However, the set of G_2-structures that are compatible with a fixed metric can be parametrized by sections of a *unit octonion bundle*.
- The relationship between the group G_2 and the octonions is of course natural - G_2 is the automorphism group of the non-associative octonion algebra.
- It has been noted [SG'17] that the choice of the G_2-structure in a fixed metric class can be regarded as choice of gauge, with gauge transformations given by octonion multiplication.
- This talk is based on the following papers:
  1. SG, *G_2-structures and octonion bundles*, Adv. Math, 2017, arXiv:1510.04226
  2. SG, *Smooth Loops and Smooth Bundles,* Adv. Math., 2021, arXiv: 2008.08120
  3. SG, *The Coulomb gauge in non-associative gauge theory,* arXiv:2303.00664
- This work has been supported by the NSF grant DMS-1811754.

Loops

Definition

A *quasigroup* $\mathbb{L}$ is a set together with the following operations $\mathbb{L} \times \mathbb{L} \longrightarrow \mathbb{L}$

1. Product $(p, q) \mapsto pq$
2. Right quotient $(p, q) \mapsto p/q$
3. Left quotient $(p, q) \mapsto q\backslash p$,

Loops

Definition

A *quasigroup* $\mathbb{L}$ is a set together with the following operations $\mathbb{L} \times \mathbb{L} \longrightarrow \mathbb{L}$

1. Product $(p, q) \mapsto pq$
2. Right quotient $(p, q) \mapsto p/q$
3. Left quotient $(p, q) \mapsto q\backslash p$,

that satisfy the following properties

$$(p/q)\, q = p \qquad q\,(q\backslash p) = p \qquad (pq)\,/q = p \qquad p\backslash\,(pq) = q.$$

Loops

Definition

A *quasigroup* $\mathbb{L}$ is a set together with the following operations $\mathbb{L} \times \mathbb{L} \longrightarrow \mathbb{L}$

1. Product $(p, q) \mapsto pq$
2. Right quotient $(p, q) \mapsto p/q$
3. Left quotient $(p, q) \mapsto q\backslash p$,

that satisfy the following properties

$$(p/q)\, q = p \qquad q\,(q\backslash p) = p \qquad (pq)\,/q = p \qquad p\backslash\,(pq) = q.$$

- A *loop* is a quasigroup with an identity element 1 (i.e. a unital quasigroup).

Loops

Definition

A *quasigroup* $\mathbb{L}$ is a set together with the following operations $\mathbb{L} \times \mathbb{L} \longrightarrow \mathbb{L}$

1. Product $(p, q) \mapsto pq$
2. Right quotient $(p, q) \mapsto p/q$
3. Left quotient $(p, q) \mapsto q \backslash p$,

that satisfy the following properties

$$(p/q)\, q = p \qquad q\,(q \backslash p) = p \qquad (pq)\,/q = p \qquad p \backslash (pq) = q.$$

- A *loop* is a quasigroup with an identity element 1 (i.e. a unital quasigroup).
- A *smooth loop* is a loop that is also a smooth manifold, with left and right product maps L_p and R_p being diffeomorphisms for every $p \in \mathbb{L}$.

Examples

Example

Any group is a loop, and conversely, any associative loop is a group.

Examples

Example

Any group is a loop, and conversely, any associative loop is a group.

Example

The set S^7 of unit octonions forms a non-associative loop. It satisfies additional identities, and is a *Moufang loop*.

Examples

Example

Any group is a loop, and conversely, any associative loop is a group.

Example

The set S^7 of unit octonions forms a non-associative loop. It satisfies additional identities, and is a *Moufang loop*.

Example

Consider the set P_n of positive-definite, symmetric real matrices. Then define a product $A \circ B$ of two such matrices given by

$$A \circ B = \left(AB^2A\right)^{\frac{1}{2}} .$$

Examples

Example

Any group is a loop, and conversely, any associative loop is a group.

Example

The set S^7 of unit octonions forms a non-associative loop. It satisfies additional identities, and is a *Moufang loop*.

Example

Consider the set P_n of positive-definite, symmetric real matrices. Then define a product $A \circ B$ of two such matrices given by

$$A \circ B = \left(AB^2A\right)^{\frac{1}{2}}.$$

This product defines a loop structure on P_n.

Examples

Example

Any group is a loop, and conversely, any associative loop is a group.

Example

The set S^7 of unit octonions forms a non-associative loop. It satisfies additional identities, and is a *Moufang loop*.

Example

Consider the set P_n of positive-definite, symmetric real matrices. Then define a product $A \circ B$ of two such matrices given by

$$A \circ B = \left(AB^2A\right)^{\frac{1}{2}} .$$

This product defines a loop structure on P_n. If $n > 1$, it is non-associative.

Pseudoautomorphisms

Definition

A *pseudoautomorphism* of a smooth loop $\mathbb{L}$ is a diffeomorphism $h : \mathbb{L} \longrightarrow \mathbb{L}$, such that there exists a diffeomorphism $h' : \mathbb{L} \longrightarrow \mathbb{L}$, known as the *partial pseudoautomorphism* corresponding to h, such that for any $p, q \in \mathbb{L}$,

$$h(pq) = h'(p)\, h(q)\,. \qquad (1)$$

Pseudoautomorphisms

Definition

A *pseudoautomorphism* of a smooth loop $\mathbb{L}$ is a diffeomorphism $h : \mathbb{L} \longrightarrow \mathbb{L}$, such that there exists a diffeomorphism $h' : \mathbb{L} \longrightarrow \mathbb{L}$, known as the *partial pseudoautomorphism* corresponding to h, such that for any $p, q \in \mathbb{L}$,

$$h(pq) = h'(p)\, h(q)\,. \quad (1)$$

- The element $h(1) \in \mathbb{L}$ is the *companion* of h', and $h = R_{h(1)} \circ h'$.

Pseudoautomorphisms

Definition

A *pseudoautomorphism* of a smooth loop $\mathbb{L}$ is a diffeomorphism $h : \mathbb{L} \longrightarrow \mathbb{L}$, such that there exists a diffeomorphism $h' : \mathbb{L} \longrightarrow \mathbb{L}$, known as the *partial pseudoautomorphism* corresponding to h, such that for any $p, q \in \mathbb{L}$,

$$h(pq) = h'(p) h(q) . \tag{1}$$

- The element $h(1) \in \mathbb{L}$ is the *companion* of h', and $h = R_{h(1)} \circ h'$.
- We also see the following property of h' with respect to quotients: $h'(p/q) = h(p)/h(q)$.

Pseudoautomorphisms

Definition

A *pseudoautomorphism* of a smooth loop $\mathbb{L}$ is a diffeomorphism $h : \mathbb{L} \longrightarrow \mathbb{L}$, such that there exists a diffeomorphism $h' : \mathbb{L} \longrightarrow \mathbb{L}$, known as the *partial pseudoautomorphism* corresponding to h, such that for any $p, q \in \mathbb{L}$,

$$h(pq) = h'(p)\, h(q)\,. \tag{1}$$

- The element $h(1) \in \mathbb{L}$ is the *companion* of h', and $h = R_{h(1)} \circ h'$.
- We also see the following property of h' with respect to quotients: $h'(p/q) = h(p)/h(q)\,.$
- The sets of pseudoautomorphisms Ψ and partial pseudoautomorphisms Ψ' are both Lie groups (at least for compact $\mathbb{L}$ they are finite-dimensional).

Pseudoautomorphisms

Definition

A *pseudoautomorphism* of a smooth loop $\mathbb{L}$ is a diffeomorphism $h : \mathbb{L} \longrightarrow \mathbb{L}$, such that there exists a diffeomorphism $h' : \mathbb{L} \longrightarrow \mathbb{L}$, known as the *partial pseudoautomorphism* corresponding to h, such that for any $p, q \in \mathbb{L}$,

$$h(pq) = h'(p)\, h(q)\,. \tag{1}$$

- The element $h(1) \in \mathbb{L}$ is the *companion* of h', and $h = R_{h(1)} \circ h'$.
- We also see the following property of h' with respect to quotients: $h'(p/q) = h(p)/h(q)\,.$
- The sets of pseudoautomorphisms Ψ and partial pseudoautomorphisms Ψ' are both Lie groups (at least for compact $\mathbb{L}$ they are finite-dimensional).
- We also see that the *automorphism* group of the loop $\mathbb{L}$ is the subgroup $H \subset \Psi$ which is the stabilizer of $1 \in \mathbb{L}$.

- We can regard $\mathbb{L}$ as a set with the action of Ψ or with action of Ψ'. In latter case, denote the Ψ'-set by $\mathbb{L}'$.

- We can regard $\mathbb{L}$ as a set with the action of Ψ or with action of Ψ'. In latter case, denote the Ψ'-set by $\mathbb{L}'$.
- From (1), we can regard the loop product as a map $\mathbb{L}' \times \mathbb{L} \longrightarrow \mathbb{L}$.

- We can regard $\mathbb{L}$ as a set with the action of Ψ or with action of Ψ'. In latter case, denote the Ψ'-set by $\mathbb{L}'$.
- From (1), we can regard the loop product as a map $\mathbb{L}' \times \mathbb{L} \longrightarrow \mathbb{L}$.

- We can regard $\mathbb{L}$ as a set with the action of Ψ or with action of Ψ'. In latter case, denote the Ψ'-set by $\mathbb{L}'$.
- From (1), we can regard the loop product as a map $\mathbb{L}' \times \mathbb{L} \longrightarrow \mathbb{L}$.

Example

Suppose $\mathbb{L} = S^7$. In this case, $\Psi\left(S^7\right) \cong \mathrm{Spin}\left(7\right)$ and $\Psi'\left(S^7\right) \cong SO\left(7\right)$. The automorphism group is G_2. The action of $SO\left(7\right)$ corresponds to the vector representation of $\mathrm{Spin}\left(7\right)$ and the 'full' action of $\mathrm{Spin}\left(7\right)$ corresponds to the spinor representation.

Tangent algebra

- Recall that for any $s \in \mathbb{L}$, we have the diffeomorphisms

$$\begin{aligned} R_s : \mathbb{L} &\longrightarrow \mathbb{L} \\ q &\longmapsto qs. \end{aligned}$$

Tangent algebra

- Recall that for any $s \in \mathbb{L}$, we have the diffeomorphisms

$$\begin{aligned} R_s : \mathbb{L} &\longrightarrow \mathbb{L} \\ q &\longmapsto qs. \end{aligned}$$

- Given a tangent vector $\xi \in T_1\mathbb{L}$, define the vector field $\rho(\xi)$ given by

$$\rho(\xi)_q = (R_q)_* \xi \tag{2}$$

at any $p \in \mathbb{L}$.

Tangent algebra

Definition

For any $\xi, \gamma \in T_1\mathbb{L}$, the p-bracket $[\cdot, \cdot]^{(p)}$ is defined as

$$[\xi, \gamma]^{(p)} = -\left(R_p^{-1}\right)_* [\rho(\xi), \rho(\gamma)]_p . \quad (3)$$

Tangent algebra

Definition

For any $\xi, \gamma \in T_1\mathbb{L}$, the p-bracket $[\cdot,\cdot]^{(p)}$ is defined as

$$[\xi,\gamma]^{(p)} = -\left(R_p^{-1}\right)_* \left[\rho\left(\xi\right), \rho\left(\gamma\right)\right]_p . \tag{3}$$

Definition

The vector space $T_1\mathbb{L}$ together with the bracket $[\cdot,\cdot]^{(p)}$ is the *tangent algebra* $\mathfrak{l}^{(p)}$ of $(\mathbb{L}, \circ_p)$.

Tangent algebra

Definition

For any $\xi, \gamma \in T_1\mathbb{L}$, the p-bracket $[\cdot,\cdot]^{(p)}$ is defined as

$$[\xi,\gamma]^{(p)} = -\left(R_p^{-1}\right)_* \left[\rho\left(\xi\right), \rho\left(\gamma\right)\right]_p . \quad (3)$$

Definition

The vector space $T_1\mathbb{L}$ together with the bracket $[\cdot,\cdot]^{(p)}$ is the *tangent algebra* $\mathfrak{l}^{(p)}$ of $(\mathbb{L}, \circ_p)$.

- Define $b : \mathbb{L} \longrightarrow \mathfrak{l} \otimes \Lambda^2\mathfrak{l}^*$ given by $p \mapsto [\cdot,\cdot]^{(p)}$. Then, in general $db|_p \neq 0$. Let $a\left(\xi,\eta,\gamma\right) = d_{\rho(\gamma)} b\left(\xi,\eta\right).$

Maurer-Cartan form

- Given $p \in \mathbb{L}$ and and $\xi \in \mathfrak{l}$, define $\theta_p \in \Omega^1(\mathfrak{l})$ via $\theta_p\left(\rho(\xi)_p\right) = \left(R_p^{-1}\right)_* \rho(\xi)_p = \xi$.

Maurer-Cartan form

- Given $p \in \mathbb{L}$ and and $\xi \in \mathfrak{l}$, define $\theta_p \in \Omega^1(\mathfrak{l})$ via $\theta_p\left(\rho(\xi)_p\right) = \left(R_p^{-1}\right)_* \rho(\xi)_p = \xi$.

Maurer-Cartan form

- Given $p \in \mathbb{L}$ and and $\xi \in \mathfrak{l}$, define $\theta_p \in \Omega^1(\mathfrak{l})$ via $\theta_p\left(\rho(\xi)_p\right) = \left(R_p^{-1}\right)_* \rho(\xi)_p = \xi.$

Theorem (SG'21)

Let $p \in \mathbb{L}$ and let $[\cdot,\cdot]^{(p)}$ be bracket on $\mathfrak{l}^{(p)}$. Then θ satisfies the following equation at p:

$$(d\theta)_p - \frac{1}{2}[\theta,\theta]^{(p)} = 0, \tag{4}$$

where $[\theta,\theta]^{(p)}$ is the bracket of $\mathbb{L}$-algebra-valued 1-forms such that for any $X, Y \in T_p\mathbb{L}$, $\frac{1}{2}[\theta,\theta]^{(p)}(X,Y) = [\theta(X),\theta(Y)]^{(p)}$.

Maurer-Cartan form

- Given $p \in \mathbb{L}$ and and $\xi \in \mathfrak{l}$, define $\theta_p \in \Omega^1(\mathfrak{l})$ via $\theta_p\left(\rho(\xi)_p\right) = \left(R_p^{-1}\right)_* \rho(\xi)_p = \xi$.

Theorem (SG'21)

Let $p \in \mathbb{L}$ and let $[\cdot,\cdot]^{(p)}$ be bracket on $\mathfrak{l}^{(p)}$. Then θ satisfies the following equation at p:

$$(d\theta)_p - \frac{1}{2}[\theta,\theta]^{(p)} = 0, \quad (4)$$

where $[\theta,\theta]^{(p)}$ is the bracket of $\mathbb{L}$-algebra-valued 1-forms such that for any $X, Y \in T_p\mathbb{L}$, $\frac{1}{2}[\theta,\theta]^{(p)}(X,Y) = [\theta(X),\theta(Y)]^{(p)}$.

- From (4) we obtain a generalization of the Jacobi identity, known as the Akivis identity:

$$\mathrm{Jac}^{(p)}(\xi,\eta,\gamma) = a_p(\xi,\eta,\gamma) + a_p(\eta,\gamma,\xi) + a_p(\gamma,\xi,\eta). \quad (5)$$

Exponential map

- We can define exponential maps via integral curves of fundamental vector fields.

Exponential map

- We can define exponential maps via integral curves of fundamental vector fields.
- Let $\xi \in \mathfrak{l}$, if $\rho(\xi)$ is complete, define $\exp_s \xi = p_\xi(1)/s$, where $p_\xi(t)$ is the maximal integral curve of $\rho(\xi)$ with $p_\xi(0) = s$.

Exponential map

- We can define exponential maps via integral curves of fundamental vector fields.
- Let $\xi \in \mathfrak{l}$, if $\rho(\xi)$ is complete, define $\exp_s \xi = p_\xi(1)/s$, where $p_\xi(t)$ is the maximal integral curve of $\rho(\xi)$ with $p_\xi(0) = s$.
- If $\mathbb{L}$ is compact, then any vector field is complete.

Exponential map

- We can define exponential maps via integral curves of fundamental vector fields.
- Let $\xi \in \mathfrak{l}$, if $\rho(\xi)$ is complete, define $\exp_s \xi = p_\xi(1)/s$, where $p_\xi(t)$ is the maximal integral curve of $\rho(\xi)$ with $p_\xi(0) = s$.
- If $\mathbb{L}$ is compact, then any vector field is complete.
- If $\mathbb{L}$ is non-compact, need power-associativity, so that powers of an element $p \in \mathbb{L}$ associate, then $p_\xi(nh) = p_\xi(h)^n$ can be defined unambiguously (Kuz'min'1971).

Exponential map

- We can define exponential maps via integral curves of fundamental vector fields.
- Let $\xi \in \mathfrak{l}$, if $\rho\left(\xi\right)$ is complete, define $\exp_s \xi = p_{\xi}\left(1\right)/s$, where $p_{\xi}\left(t\right)$ is the maximal integral curve of $\rho\left(\xi\right)$ with $p_{\xi}\left(0\right) = s$.
- If $\mathbb{L}$ is compact, then any vector field is complete.
- If $\mathbb{L}$ is non-compact, need power-associativity, so that powers of an element $p \in \mathbb{L}$ associate, then $p_{\xi}\left(nh\right) = p_{\xi}\left(h\right)^n$ can be defined unambiguously (Kuz'min'1971).
- $\exp_s : \mathfrak{l} \longrightarrow \mathbb{L}$ is a local diffeomorphism from neighborhood of $0 \in \mathfrak{l}$ to a neighborhood of $1 \in \mathbb{L}$.

Exponential map

- We can define exponential maps via integral curves of fundamental vector fields.
- Let $\xi \in \mathfrak{l}$, if $\rho(\xi)$ is complete, define $\exp_s \xi = p_\xi(1)/s$, where $p_\xi(t)$ is the maximal integral curve of $\rho(\xi)$ with $p_\xi(0) = s$.
- If $\mathbb{L}$ is compact, then any vector field is complete.
- If $\mathbb{L}$ is non-compact, need power-associativity, so that powers of an element $p \in \mathbb{L}$ associate, then $p_\xi(nh) = p_\xi(h)^n$ can be defined unambiguously (Kuz'min'1971).
- $\exp_s : \mathfrak{l} \longrightarrow \mathbb{L}$ is a local diffeomorphism from neighborhood of $0 \in \mathfrak{l}$ to a neighborhood of $1 \in \mathbb{L}$.
- Under additional assumptions on $\mathbb{L}$ (left power-associative), $\exp_s$ is independent on s.

- For a fixed $\xi \in \mathfrak{l}$ and $s \in \mathbb{L}$, consider the equation for $\mathfrak{l}$-valued quantity η
$$\frac{d\eta}{dt} = [\xi, \eta]^{\exp_s(t\xi)s} \, .$$

- For a fixed $\xi \in \mathfrak{l}$ and $s \in \mathbb{L}$, consider the equation for $\mathfrak{l}$-valued quantity η

$$\frac{d\eta}{dt} = [\xi, \eta]^{\exp_s(t\xi)s}.$$

- This is a homogeneous linear first-order ODE, so for all $t \in I$ there exists an evolution operator $U_{\xi}^{(s)}(t) \in GL(\mathfrak{l})$, with $U_{\xi}^{(s)}(0) = \mathrm{id}_{\mathfrak{l}}$, such that

$$\eta(t) = U_{\xi}^{(s)}(t)\,\eta_0. \tag{6}$$

- For a fixed $\xi \in \mathfrak{l}$ and $s \in \mathbb{L}$, consider the equation for $\mathfrak{l}$-valued quantity η
$$\frac{d\eta}{dt} = [\xi, \eta]^{\exp_s(t\xi)s}.$$
- This is a homogeneous linear first-order ODE, so for all $t \in I$ there exists an evolution operator $U_\xi^{(s)}(t) \in GL(\mathfrak{l})$, with $U_\xi^{(s)}(0) = \mathrm{id}_{\mathfrak{l}}$, such that
$$\eta(t) = U_\xi^{(s)}(t)\,\eta_0. \quad (6)$$
- In the Lie group setting, the bracket is constant, so in that case $U_\xi^{(s)}(t) = e^{\mathrm{ad}_{t\xi}} = \mathrm{Ad}_{\exp(t\xi)}$.

Loop bundles

- Let M be a smooth manifold with a Ψ-principal bundle $\mathcal{P}$. Recall that if S is a set with an action of Ψ on it, then we can define an associated bundle $\mathcal{P} \times_{\Psi} S$, with sections being in an 1-1 correspondence with equivariant maps $\mathcal{P} \longrightarrow S$. Define the following bundles:

Loop bundles

- Let M be a smooth manifold with a Ψ-principal bundle $\mathcal{P}$. Recall that if S is a set with an action of Ψ on it, then we can define an associated bundle $\mathcal{P} \times_{\Psi} S$, with sections being in an 1-1 correspondence with equivariant maps $\mathcal{P} \longrightarrow S$. Define the following bundles:

Loop bundles

- Let M be a smooth manifold with a Ψ-principal bundle $\mathcal{P}$. Recall that if S is a set with an action of Ψ on it, then we can define an associated bundle $\mathcal{P}\times_{\Psi} S$, with sections being in an 1-1 correspondence with equivariant maps $\mathcal{P}\longrightarrow S$. Define the following bundles: ($h\in\Psi$, and $\mathbb{L}'$ is the set $\mathbb{L}$ equipped the action of Ψ').

Bundle	**Equivariant map**	**Equivariance property**
$\mathcal{P}$	$k:\mathcal{P}\longrightarrow\Psi$	$k_{ph}=h^{-1}k_p$
$\mathcal{Q}'=\mathcal{P}\times_{\Psi'}\mathbb{L}'$	$q:\mathcal{P}\longrightarrow\mathbb{L}'$	$q_{ph}=(h')^{-1}q_p$
$\mathcal{Q}=\mathcal{P}\times_{\Psi}\mathbb{L}$	$r:\mathcal{P}\longrightarrow\mathbb{L}$	$r_{ph}=h^{-1}(r_p)$
$\mathcal{A}=\mathcal{P}\times_{\Psi'_*}\mathfrak{l}$	$\eta:\mathcal{P}\longrightarrow\mathfrak{l}$	$\eta_{ph}=(h')_*^{-1}\eta_p$

- Suppose $A \in C^{\infty}(\mathcal{P}, \mathbb{L}')$ and $s \in C^{\infty}(\mathcal{P}, \mathbb{L})$, then if both A and s are equivariant, so is $As \in C^{\infty}(\mathcal{P}, \mathbb{L})$.

- Suppose $A \in C^{\infty}(\mathcal{P}, \mathbb{L}')$ and $s \in C^{\infty}(\mathcal{P}, \mathbb{L})$, then if both A and s are equivariant, so is $As \in C^{\infty}(\mathcal{P}, \mathbb{L})$.
- Conversely, if $r \in C^{\infty}(\mathcal{P}, \mathbb{L})$ is equivariant, we can write $r = As$, for equivariant $A = r/s \in C^{\infty}(\mathcal{P}, \mathbb{L}')$.

- Suppose $A \in C^{\infty}(\mathcal{P}, \mathbb{L}')$ and $s \in C^{\infty}(\mathcal{P}, \mathbb{L})$, then if both A and s are equivariant, so is $As \in C^{\infty}(\mathcal{P}, \mathbb{L})$.
- Conversely, if $r \in C^{\infty}(\mathcal{P}, \mathbb{L})$ is equivariant, we can write $r = As$, for equivariant $A = r/s \in C^{\infty}(\mathcal{P}, \mathbb{L}')$.
- Choose a defining equivariant map $s \in C^{\infty}(\mathcal{P}, \mathbb{L})$. The choice is arbitrary, but it allows to compare equivariant $\mathbb{L}$-valued maps.

- Suppose $A \in C^{\infty}(\mathcal{P}, \mathbb{L}')$ and $s \in C^{\infty}(\mathcal{P}, \mathbb{L})$, then if both A and s are equivariant, so is $As \in C^{\infty}(\mathcal{P}, \mathbb{L})$.
- Conversely, if $r \in C^{\infty}(\mathcal{P}, \mathbb{L})$ is equivariant, we can write $r = As$, for equivariant $A = r/s \in C^{\infty}(\mathcal{P}, \mathbb{L}')$.
- Choose a defining equivariant map $s \in C^{\infty}(\mathcal{P}, \mathbb{L})$. The choice is arbitrary, but it allows to compare equivariant $\mathbb{L}$-valued maps.
- Given s, easy to show that corresponding maps b_s and a_s are also equivariant.

Connections and Torsion

- Suppose the principal Ψ-bundle $\mathcal{P}$ has a principal connection given by

$$T\mathcal{P} = \mathcal{HP} \oplus \mathcal{VP}$$

and let $\omega : T\mathcal{P} \longrightarrow \mathfrak{p}$ be the corresponding connection 1-form, where $\mathfrak{p}$ is the Lie algebra of Ψ.

Connections and Torsion

- Suppose the principal Ψ-bundle $\mathcal{P}$ has a principal connection given by

$$T\mathcal{P} = \mathcal{HP} \oplus \mathcal{VP}$$

and let $\omega : T\mathcal{P} \longrightarrow \mathfrak{p}$ be the corresponding connection 1-form, where $\mathfrak{p}$ is the Lie algebra of Ψ.

- Recall that given an equivariant map $f : \mathcal{P} \longrightarrow S$, the covariant derivative is defined as

$$d^{\omega} f := f_* \circ \mathrm{proj}_{\mathcal{H}} : T\mathcal{P} \longrightarrow \mathcal{HP} \longrightarrow TS. \tag{7}$$

Connections and Torsion

- Suppose the principal Ψ-bundle $\mathcal{P}$ has a principal connection given by

$$T\mathcal{P} = \mathcal{HP} \oplus \mathcal{VP}$$

and let $\omega : T\mathcal{P} \longrightarrow \mathfrak{p}$ be the corresponding connection 1-form, where $\mathfrak{p}$ is the Lie algebra of Ψ.

- Recall that given an equivariant map $f : \mathcal{P} \longrightarrow S$, the covariant derivative is defined as

$$d^{\omega} f := f_{*} \circ \mathrm{proj}_{\mathcal{H}} : T\mathcal{P} \longrightarrow \mathcal{HP} \longrightarrow TS. \tag{7}$$

- Equivalently,

$$d^{\omega} f = df + \omega \cdot f$$

where $\cdot$ denotes the infinitesimal action of $\mathfrak{p}$ on S.

- Let $s \in C^{\infty}(\mathcal{P}, \mathbb{L})$ be equivariant and let ω be a connection on $\mathcal{P}$.

- Let $s \in C^{\infty}(\mathcal{P}, \mathbb{L})$ be equivariant and let ω be a connection on $\mathcal{P}$.

- Let $s \in C^{\infty}(\mathcal{P}, \mathbb{L})$ be equivariant and let ω be a connection on $\mathcal{P}$.

Definition

The *torsion* $T^{(s,\omega)}$ of s and ω is a horizontal $\mathfrak{l}$-valued 1-form on $\mathcal{P}$ given by

$$T^{(s,\omega)} = s^{*}\theta \circ \mathrm{proj}_{\mathcal{H}} \tag{8}$$

Equivalently, at $p \in \mathcal{P}$, we have

$$\left. T^{(s,\omega)} \right|_{p} = \left(R_{s_p}^{-1} \right)_{*} d^{\omega} s|_{p} \,. \tag{9}$$

- Equivariant horizontal (i.e. *basic*) forms on $\mathcal{P}$ give rise to sections of corresponding associated bundles over the base manifold M.

- Equivariant horizontal (i.e. *basic*) forms on $\mathcal{P}$ give rise to sections of corresponding associated bundles over the base manifold M.
- Let us now switch perspective, and work in terms of sections of bundles.

- Equivariant horizontal (i.e. *basic*) forms on $\mathcal{P}$ give rise to sections of corresponding associated bundles over the base manifold M.
- Let us now switch perspective, and work in terms of sections of bundles.
- We will consider s to be a smooth section of the bundle $\mathcal{Q}$, so that we will say $s \in \Gamma(\mathcal{Q})$, and will refer to it as the *defining section*.

- Equivariant horizontal (i.e. *basic*) forms on $\mathcal{P}$ give rise to sections of corresponding associated bundles over the base manifold M.
- Let us now switch perspective, and work in terms of sections of bundles.
- We will consider s to be a smooth section of the bundle $\mathcal{Q}$, so that we will say $s \in \Gamma(\mathcal{Q})$, and will refer to it as the *defining section*.
- Also consider sections $A \in \Gamma(\mathcal{Q}')$, which admit the action of Ψ'.

- Equivariant horizontal (i.e. *basic*) forms on $\mathcal{P}$ give rise to sections of corresponding associated bundles over the base manifold M.
- Let us now switch perspective, and work in terms of sections of bundles.
- We will consider s to be a smooth section of the bundle $\mathcal{Q}$, so that we will say $s \in \Gamma\left(\mathcal{Q}\right)$, and will refer to it as the *defining section*.
- Also consider sections $A \in \Gamma\left(\mathcal{Q}'\right)$, which admit the action of Ψ'.
- The product on elements of $\mathbb{L}'$ and $\mathbb{L}$, then carries over to sections of bundles, so that we have a product $\Gamma\left(\mathcal{Q}'\right) \times \Gamma\left(\mathcal{Q}\right) \longrightarrow \Gamma\left(\mathcal{Q}\right)$.

- Equivariant horizontal (i.e. *basic*) forms on $\mathcal{P}$ give rise to sections of corresponding associated bundles over the base manifold M.
- Let us now switch perspective, and work in terms of sections of bundles.
- We will consider s to be a smooth section of the bundle $\mathcal{Q}$, so that we will say $s \in \Gamma(\mathcal{Q})$, and will refer to it as the *defining section*.
- Also consider sections $A \in \Gamma(\mathcal{Q}')$, which admit the action of Ψ'.
- The product on elements of $\mathbb{L}'$ and $\mathbb{L}$, then carries over to sections of bundles, so that we have a product $\Gamma(\mathcal{Q}') \times \Gamma(\mathcal{Q}) \longrightarrow \Gamma(\mathcal{Q})$.
- The connection ω on $\mathcal{P}$ then induces connections on the associated bundles and correspondingly, covariant derivatives on sections of these bundles.

- Equivariant horizontal (i.e. *basic*) forms on $\mathcal{P}$ give rise to sections of corresponding associated bundles over the base manifold M.
- Let us now switch perspective, and work in terms of sections of bundles.
- We will consider s to be a smooth section of the bundle $\mathcal{Q}$, so that we will say $s \in \Gamma\left(\mathcal{Q}\right)$, and will refer to it as the *defining section*.
- Also consider sections $A \in \Gamma\left(\mathcal{Q}'\right)$, which admit the action of Ψ'.
- The product on elements of $\mathbb{L}'$ and $\mathbb{L}$, then carries over to sections of bundles, so that we have a product $\Gamma\left(\mathcal{Q}'\right) \times \Gamma\left(\mathcal{Q}\right) \longrightarrow \Gamma\left(\mathcal{Q}\right)$.
- The connection ω on $\mathcal{P}$ then induces connections on the associated bundles and correspondingly, covariant derivatives on sections of these bundles.
- The torsion $T^{(s,\omega)}$, as defined earlier, was a horizontal and equivariant 1-form on $\mathcal{P}$ with values in $\mathfrak{l}$, so it uniquely corresponds to a 1-form on M with values in the bundle $\mathcal{A}$, i.e., now we will consider $T^{(s,\omega)} \in \Omega^1\left(\mathcal{A}\right)$.

Non-associative gauge theory

Definition

A *non-associative gauge theory* is defined by the following objects:

1. A smooth loop $\mathbb{L}$ with a finite-dimensional pseudoautomorphism Lie group Ψ and tangent algebra $\mathfrak{l}$ at identity.

Non-associative gauge theory

Definition

A *non-associative gauge theory* is defined by the following objects:

1. A smooth loop $\mathbb{L}$ with a finite-dimensional pseudoautomorphism Lie group Ψ and tangent algebra $\mathfrak{l}$ at identity.
2. A smooth manifold M with a principal Ψ-bundle $\mathcal{P}$, and associated bundles $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{A}$, with fibers $\mathbb{L}$, $\mathbb{L}'$, and $\mathfrak{l}$, respectively.

Non-associative gauge theory

Definition

A *non-associative gauge theory* is defined by the following objects:

1. A smooth loop $\mathbb{L}$ with a finite-dimensional pseudoautomorphism Lie group Ψ and tangent algebra $\mathfrak{l}$ at identity.
2. A smooth manifold M with a principal Ψ-bundle $\mathcal{P}$, and associated bundles $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{A}$, with fibers $\mathbb{L}$, $\mathbb{L}'$, and $\mathfrak{l}$, respectively.
3. A *configuration* (s, ω), where $s \in \Gamma(\mathcal{Q})$ is a defining section and ω is a connection on $\mathcal{P}$. Each configuration carries torsion $T^{(s,\omega)} \in \Omega^1(\mathcal{A})$.

Non-associative gauge theory

Definition

A *non-associative gauge theory* is defined by the following objects:

1. A smooth loop $\mathbb{L}$ with a finite-dimensional pseudoautomorphism Lie group Ψ and tangent algebra $\mathfrak{l}$ at identity.
2. A smooth manifold M with a principal Ψ-bundle $\mathcal{P}$, and associated bundles $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{A}$, with fibers $\mathbb{L}$, $\mathbb{L}'$, and $\mathfrak{l}$, respectively.
3. A *configuration* (s, ω), where $s \in \Gamma(\mathcal{Q})$ is a defining section and ω is a connection on $\mathcal{P}$. Each configuration carries torsion $T^{(s,\omega)} \in \Omega^1(\mathcal{A})$.

Non-associative gauge theory

Definition

A *non-associative gauge theory* is defined by the following objects:

1. A smooth loop $\mathbb{L}$ with a finite-dimensional pseudoautomorphism Lie group Ψ and tangent algebra $\mathfrak{l}$ at identity.
2. A smooth manifold M with a principal Ψ-bundle $\mathcal{P}$, and associated bundles $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{A}$, with fibers $\mathbb{L}$, $\mathbb{L}'$, and $\mathfrak{l}$, respectively.
3. A *configuration* (s, ω), where $s \in \Gamma(\mathcal{Q})$ is a defining section and ω is a connection on $\mathcal{P}$. Each configuration carries torsion $T^{(s,\omega)} \in \Omega^1(\mathcal{A})$.

Definition

A *loop gauge transformation* is a transformation of the defining section s by left multiplication by a section $A \in \Gamma(\mathcal{Q}')$, such that $s \mapsto As$, and hence $T^{(s,\omega)} \mapsto T^{(As,\omega)}$.

Lemma (SG'23)

Suppose $T^{(s,\omega)}$ *is the torsion with respect to a defining section* $s \in \Gamma(\mathcal{Q})$ *and a connection* ω*. Suppose* $A_t = \exp_s(t\xi) \in \Gamma(\mathcal{Q}')$, *then*

$$\frac{d}{dt} T^{(A_t s,\omega)} = \left[\xi, T^{(A_t s,\omega)}\right]^{(A_t s)} + d^{\omega}\xi. \tag{10}$$

Lemma (SG'23)

Suppose $T^{(s,\omega)}$ is the torsion with respect to a defining section $s \in \Gamma(\mathcal{Q})$ and a connection ω. Suppose $A_t = \exp_s(t\xi) \in \Gamma(\mathcal{Q}')$, then

$$\frac{d}{dt} T^{(A_t s,\omega)} = \left[\xi, T^{(A_t s,\omega)}\right]^{(A_t s)} + d^{\omega}\xi. \tag{10}$$

In particular, we get

$$\begin{aligned} T^{((\exp_s t\xi)s,\omega)} &= U_{\xi}^{(s)}(t)\, T^{(s,\omega)} \\ &\quad + U_{\xi}^{(s)}(t) \left(\int_0^1 U_{\xi}^{(s)}(\tau)^{-1} d\tau \right) d^{\omega}\xi. \end{aligned} \tag{11}$$

Energy functional

- Suppose M is compact with a metric g. Define the functional

$$\mathcal{E}_{\omega}(s) = \int_M \left|T^{(s,\omega)}\right|^2 \mathrm{vol}_g, \tag{12}$$

where $|\cdot|$ is a combination of the metric g on M and an inner product on sections of $\mathcal{A}$.

Energy functional

- Suppose M is compact with a metric g. Define the functional

$$\mathcal{E}_{\omega}\left(s\right)=\int_{M}\left|T^{\left(s,\omega\right)}\right|^{2}\mathrm{vol}_{g}, \tag{12}$$

where $\left|\cdot\right|$ is a combination of the metric g on M and an inner product on sections of $\mathcal{A}$.

Energy functional

- Suppose M is compact with a metric g. Define the functional

$$\mathcal{E}_{\omega}\left(s\right)=\int_{M}\left|T^{(s,\omega)}\right|^{2}\mathrm{vol}_{g}, \tag{12}$$

where $|\cdot|$ is a combination of the metric g on M and an inner product on sections of $\mathcal{A}$.

Theorem (SG'21)

Suppose $\mathbb{L}$ is a semisimple Moufang loop, and suppose the inner product on $\mathcal{A}$ is given by the Killing form, then the critical points of the functional (12) with respect to deformations of the defining section s are those for which

$$\left(d^{\omega}\right)^{*}T^{(s,\omega)}=0. \tag{13}$$

Theorem (SG'23)

Suppose $\mathbb{L}$ *is a smooth compact loop with tangent algebra* $\mathfrak{l}$ *and pseudoautomorphism group* Ψ.

Theorem (SG'23)

Suppose $\mathbb{L}$ is a smooth compact loop with tangent algebra $\mathfrak{l}$ and pseudoautomorphism group Ψ. Let (M, g) be a closed, smooth Riemannian manifold of dimension $n \geq 2$, and let $\mathcal{P}$ be a Ψ-principal bundle over M with a smooth connection ω.

Theorem (SG'23)

Suppose $\mathbb{L}$ *is a smooth compact loop with tangent algebra* $\mathfrak{l}$ *and pseudoautomorphism group* Ψ. *Let* (M, g) *be a closed, smooth Riemannian manifold of dimension* $n \geq 2$, *and let* $\mathcal{P}$ *be a* Ψ*-principal bundle over* M *with a smooth connection* ω. *Suppose* $k \geq 0$ *and* $r \geq 0$ *such that* $(k-1)\,r \geq n$. *Then, there exist constants* $\delta \in (0, 1]$ *and* $K \in (0, \infty)$, *such that if* $s \in \Gamma(\mathcal{Q})$ *is a smooth defining section for which*

$$\left\| T^{(s,\omega)} \right\|_{W^{k-1,r}} < \delta,$$

Theorem (SG'23)

Suppose $\mathbb{L}$ *is a smooth compact loop with tangent algebra* $\mathfrak{l}$ *and pseudoautomorphism group* Ψ. *Let* (M, g) *be a closed, smooth Riemannian manifold of dimension* $n \geq 2$, *and let* $\mathcal{P}$ *be a* Ψ*-principal bundle over* M *with a smooth connection* ω. *Suppose* $k \geq 0$ *and* $r \geq 0$ *such that* $(k-1)\, r \geq n$. *Then, there exist constants* $\delta \in (0, 1]$ *and* $K \in (0, \infty)$, *such that if* $s \in \Gamma(\mathcal{Q})$ *is a smooth defining section for which*

$$\left\| T^{(s,\omega)} \right\|_{W^{k-1,r}} < \delta,$$

then there exists a smooth $A \in \Gamma(\mathcal{Q}')$, *such that*

$$(d^{\omega})^{*}\, T^{(As,\omega)} = 0$$

Theorem (SG'23)

Suppose $\mathbb{L}$ *is a smooth compact loop with tangent algebra* $\mathfrak{l}$ *and pseudoautomorphism group* Ψ. *Let* (M, g) *be a closed, smooth Riemannian manifold of dimension* $n \geq 2$, *and let* $\mathcal{P}$ *be a* Ψ*-principal bundle over* M *with a smooth connection* ω. *Suppose* $k \geq 0$ *and* $r \geq 0$ *such that* $(k-1)\, r \geq n$. *Then, there exist constants* $\delta \in (0, 1]$ *and* $K \in (0, \infty)$, *such that if* $s \in \Gamma(\mathcal{Q})$ *is a smooth defining section for which*

$$\left\| T^{(s,\omega)} \right\|_{W^{k-1,r}} < \delta,$$

then there exists a smooth $A \in \Gamma(\mathcal{Q}')$, *such that*

$$(d^{\omega})^{*}\, T^{(As,\omega)} = 0$$

and

$$\left\| T^{(As,\omega)} \right\|_{W^{k-1,r}} < K \left\| T^{(s,\omega)} \right\|_{W^{k-1,r}} \left(1 + \left\| T^{(s,\omega)} \right\|_{W^{k-1,r}}^{k-1} \right). \tag{14}$$

Outline of the proof

- The idea is to use the Implicit Function Theorem for Banach Spaces.

Outline of the proof

- The idea is to use the Implicit Function Theorem for Banach Spaces.
- Let $K = \ker \Delta^{(\omega)}$, then restricting $\xi \in K^{\perp}$, define

$$G : W^{(k-1),r}\left(T^{*}M \otimes \mathcal{A}\right) \times \left(K^{\perp} \cap W^{k,r}\left(\mathcal{A}\right)\right) \longrightarrow K^{\perp} \cap W_{A_1}^{(k-2),r}\left(\mathcal{A}\right).$$

by

$$\begin{aligned} G\left(a,\xi\right) &= \left(d^{\omega}\right)^{*}\left(U_{\xi}^{(s)}\left(a - T^{(s,\omega)}\right) + T^{\left(\left(\exp_{s}\xi\right)s,\omega\right)}\right) \\ &= \left(d^{\omega}\right)^{*}\left(U_{\xi}^{(s)}a + U_{\xi}^{(s)}\left(\int_{0}^{1} U_{\xi}^{(s)}\left(\tau\right)^{-1} d\tau\right) d^{\omega}\xi\right). \end{aligned} \quad (15)$$

- Consider the differential of G at $(a,\xi)=0$ in the direction $(b,\eta)\in W^{(k-1),r}(T^*M\otimes\mathcal{A})\times\left(K^{\perp}\cap W^{k,r}(\mathcal{A})\right)$:

$$DG|_{(0,0)}(b,\eta)=(d^{\omega})^{*}b+(d^{\omega})^{*}d^{\omega}\eta.$$

- Consider the differential of G at $(a, \xi) = 0$ in the direction $(b, \eta) \in W^{(k-1),r}(T^*M \otimes \mathcal{A}) \times \left(K^{\perp} \cap W^{k,r}(\mathcal{A})\right)$:

$$DG|_{(0,0)}(b, \eta) = (d^{\omega})^* b + (d^{\omega})^* d^{\omega} \eta.$$

- IFT gives us a unique smooth map $\xi(a)$ for any $\|a\|_{W^{(k-1),r}} < \delta$, such that $G(a, \xi(a)) = 0$.

- Consider the differential of G at $(a, \xi) = 0$ in the direction $(b, \eta) \in W^{(k-1),r}(T^*M \otimes \mathcal{A}) \times \left(K^{\perp} \cap W^{k,r}(\mathcal{A})\right)$:

$$DG|_{(0,0)}(b, \eta) = (d^{\omega})^* b + (d^{\omega})^* d^{\omega} \eta.$$

- IFT gives us a unique smooth map $\xi(a)$ for any $\|a\|_{W^{(k-1),r}} < \delta$, such that $G(a, \xi(a)) = 0$.
- Setting $a = T^{(s,\omega)}$, gives us the desired result with $A = \exp_s(\xi)$.

- Consider the differential of G at $(a, \xi) = 0$ in the direction $(b, \eta) \in W^{(k-1),r}(T^*M \otimes \mathcal{A}) \times \left(K^{\perp} \cap W^{k,r}(\mathcal{A})\right)$:

$$DG|_{(0,0)}(b, \eta) = (d^{\omega})^* b + (d^{\omega})^* d^{\omega} \eta.$$

- IFT gives us a unique smooth map $\xi(a)$ for any $\|a\|_{W^{(k-1),r}} < \delta$, such that $G(a, \xi(a)) = 0$.
- Setting $a = T^{(s,\omega)}$, gives us the desired result with $A = \exp_s(\xi)$.
- The smoothness of A follows by elliptic regularity.

G_2-geometry

- Consider $\mathbb{L} = S^7$ - the Moufang loop of unit octonions. Then we have the following correspondence:

G_2-geometry

- Consider $\mathbb{L} = S^7$ - the Moufang loop of unit octonions. Then we have the following correspondence:

G_2-geometry

- Consider $\mathbb{L} = S^7$ - the Moufang loop of unit octonions. Then we have the following correspondence:

Object	Loops	Octonions
Pseudoautomorphism group	Ψ	$\mathrm{Spin}(7)$
Partial pseudoautomorphism group	Ψ'	$SO(7)$
Automorphism group	H	G_2
Lie algebra of Ψ	$\mathfrak{p}$	$\mathfrak{so}(7)$
Loop with full action of Ψ	$\mathbb{L}$	$U\mathbb{O} \subset S_7$
Loop with partial action of Ψ	$\mathbb{L}'$	$U\mathbb{O}' \subset V_1 \oplus V_7$
Tangent algebra	$\mathfrak{l}$	$\mathrm{Im}\,\mathbb{O} \cong V_7 \cong \mathbb{R}^7$

G_2-geometry

- Consider $\mathbb{L} = S^7$ - the Moufang loop of unit octonions. Then we have the following correspondence:

Object	**Loops**	**Octonions**
Pseudoautomorphism group	Ψ	$\mathrm{Spin}(7)$
Partial pseudoautomorphism group	Ψ'	$SO(7)$
Automorphism group	H	G_2
Lie algebra of Ψ	$\mathfrak{p}$	$\mathfrak{so}(7)$
Loop with full action of Ψ	$\mathbb{L}$	$U\mathbb{O} \subset S_7$
Loop with partial action of Ψ	$\mathbb{L}'$	$U\mathbb{O}' \subset V_1 \oplus V_7$
Tangent algebra	$\mathfrak{l}$	$\mathrm{Im}\,\mathbb{O} \cong V_7 \cong \mathbb{R}^7$

- Here S_7 and V_7 are the 8-dim "spinor" and 7-dimensional "vector" representations of $\mathrm{Spin}(7)$, respectively.

- Suppose M is 7-dimensional compact smooth manifold that admits G_2-structures. Then

- Suppose M is 7-dimensional compact smooth manifold that admits G_2-structures. Then

- Suppose M is 7-dimensional compact smooth manifold that admits G_2-structures. Then

Loop bundles	G_2-**geometry**
$\mathcal{P}$	Spin structure: principal $\mathrm{Spin}\,(7)$-bundle over M
$\mathcal{Q}' = \mathcal{P} \times_{\Psi'} \mathbb{L}'$	Unit octonion bundle $U\mathbb{O}M \subset \Omega^0\,(M) \oplus TM$
$\mathcal{Q} = \mathcal{P} \times_{\Psi} \mathbb{L}$	Unit spinor bundle $U\mathcal{S}$
$\mathcal{A} = \mathcal{P} \times_{\Psi'_*} \mathfrak{l}$	Bundle of imaginary octonions: TM

- A G_2-structure φ_s may be defined by a unit spinor, i.e. a section of $s \in \mathcal{Q}$.

- A G_2-structure φ_s may be defined by a unit spinor, i.e. a section of $s \in \mathcal{Q}$.
- Then, the torsion $T^{(\xi)}$ of φ_ξ is given by

$$\nabla^S_X \xi = T^{(\xi)}_X \cdot \xi, \tag{16}$$

- A G_2-structure φ_s may be defined by a unit spinor, i.e. a section of $s \in \mathcal{Q}$.
- Then, the torsion $T^{(\xi)}$ of φ_ξ is given by

$$\nabla^S_X \xi = T^{(\xi)}_X \cdot \xi, \tag{16}$$

- We see that the torsion $T^{(\xi)}$ of the G_2-structure φ_ξ precisely corresponds to the torsion $T^{(\xi,\nabla)}$ of the section ξ with respect to the Levi-Civita connection ∇.

- A G_2-structure φ_s may be defined by a unit spinor, i.e. a section of $s \in \mathcal{Q}$.
- Then, the torsion $T^{(\xi)}$ of φ_ξ is given by

$$\nabla^S_X \xi = T^{(\xi)}_X \cdot \xi, \tag{16}$$

- We see that the torsion $T^{(\xi)}$ of the G_2-structure φ_ξ precisely corresponds to the torsion $T^{(\xi,\nabla)}$ of the section ξ with respect to the Levi-Civita connection ∇.
- Given a unit octonion section $A \in \Gamma\left(U\mathbb{O}M\right)$, $A \cdot \xi$ is again a unit spinor which defines a G_2-structure $\varphi_{A\cdot\xi}$.

- A G_2-structure φ_s may be defined by a unit spinor, i.e. a section of $s \in \mathcal{Q}$.
- Then, the torsion $T^{(\xi)}$ of φ_ξ is given by

$$\nabla^S_X \xi = T^{(\xi)}_X \cdot \xi, \tag{16}$$

- We see that the torsion $T^{(\xi)}$ of the G_2-structure φ_ξ precisely corresponds to the torsion $T^{(\xi,\nabla)}$ of the section ξ with respect to the Levi-Civita connection ∇.
- Given a unit octonion section $A \in \Gamma\left(U\mathbb{O}M\right)$, $A \cdot \xi$ is again a unit spinor which defines a G_2-structure $\varphi_{A\cdot\xi}$.
- Considering both A and ξ as octonions in $U\mathbb{O}'$ and $U\mathbb{O}$, respectively, this is just octonion multiplication $A\xi$.

Theorem (SG'23)

Suppose M is a closed 7-dimensional manifold with a smooth G_2-structure φ with torsion T with respect to the Levi-Civita connection ∇. Also, suppose k is a positive integer and p is a positive real number such that $kp > 7$. Then, there exist constants $\delta \in (0,1]$ and $K \in (0,\infty)$, such that if T satisfies

$$\|T\|_{W^{k,p}} < \delta,$$

then there exists a smooth section $V \in \Gamma(U\mathbb{O}M)$, such that

$$\operatorname{div} T^{(V)} = 0$$

and

$$\left\| T^{(V)} \right\|_{W^{k,p}} < K \|T\|_{W^{k,p}} \left(1 + \|T\|_{W^{k,p}}^k\right). \tag{17}$$

- If we choose $p = 2$ to work with Hilbert spaces, then for a smooth section V, we need $k \geq 4$, so the condition on T is to be sufficiently small in the $W^{4,2}$-norm.

Concluding remarks

- These ideas show that some well-known results and techniques from classical gauge theory can be reinterpreted and adapted in a non-associative setting.

Concluding remarks

- These ideas show that some well-known results and techniques from classical gauge theory can be reinterpreted and adapted in a non-associative setting.
- It will be interesting to see what other gauge theory concepts have an analog, since any such advances will give immediate results in G_2-geometry.

Concluding remarks

- These ideas show that some well-known results and techniques from classical gauge theory can be reinterpreted and adapted in a non-associative setting.
- It will be interesting to see what other gauge theory concepts have an analog, since any such advances will give immediate results in G_2-geometry.
- It is also interesting to see what other structures in differential geometry can be obtained by looking at loop bundles and their corresponding gauge theories.

Concluding remarks

- These ideas show that some well-known results and techniques from classical gauge theory can be reinterpreted and adapted in a non-associative setting.
- It will be interesting to see what other gauge theory concepts have an analog, since any such advances will give immediate results in G_2-geometry.
- It is also interesting to see what other structures in differential geometry can be obtained by looking at loop bundles and their corresponding gauge theories.
- Aspects of the presented theory may also carry over to homogeneous or parallelizable manifolds.