Thibault Langlais — BRIDGES Workshop: Special geometries and gauge theories.

Spectral theory on twisted connected sums

[based on arxiv 2301.03513]

1) Motivation

M-theory: quantum gravity (11D)
Low-energy EFT $\to$ bosonic fields: g metric, C 3-form.
Possible vacua: $\mathbb{R}^{3,1}\times M^7$, $(M,\langle\varphi\rangle)$ G_2-manifold, $\langle C\rangle\in\mathcal{H}^3(M)$.

Principle $(M^7,\langle\varphi\rangle,\langle C\rangle)$ determine low energy physics in 4D.

e.g. (i)

$b^k(M)$	# massless fields
eigenvalues on Δ on (co-closed) q-forms	masses^2 of the fields

Deformations of $\langle\varphi\rangle \longleftrightarrow$ varying the continuous parameters in EFT

$\hookrightarrow$ $\mathcal{M}_{G_2}$ G_2-moduli space $\longrightarrow T_\varphi\mathcal{M}=\mathcal{H}^3(M)$, $|\delta\varphi|^2=\dfrac{\|\delta\varphi\|^2_{L^2}}{\mathrm{Vol}(\varphi)}$

Swampland distance conjecture $\mathcal{M}$ moduli space, $p\in\mathcal{M}$.
If $q\in\mathcal{M}$ is moving towards an infinite-distance limit, the EFT at q has an infinite tower of states becoming light.

$$m(q)=e^{-\alpha d(p,q)}m(p),\quad \alpha>0,\quad d(p,q)\longrightarrow\infty.$$

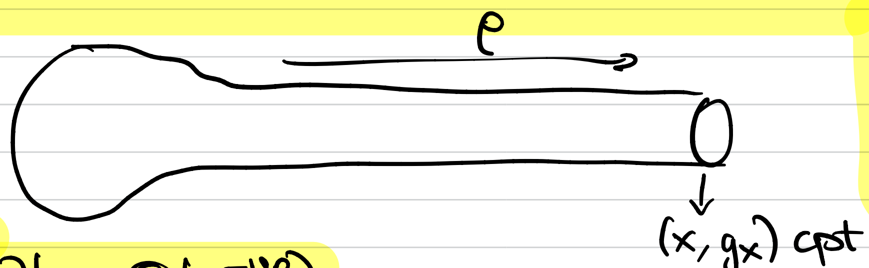
[Vafa '05, Ooguri-Vafa '06]

Goal Study this for G_2-manifolds $\to$ metric on $\mathcal{M}$; $\to$ $\boxed{\lambda_n(\varphi)}$

2) Twisted connected sums [Kovalev '03, CHNP '13, Nordström '18 (extra)]

Building blocks

EAC mfd (Z,g)



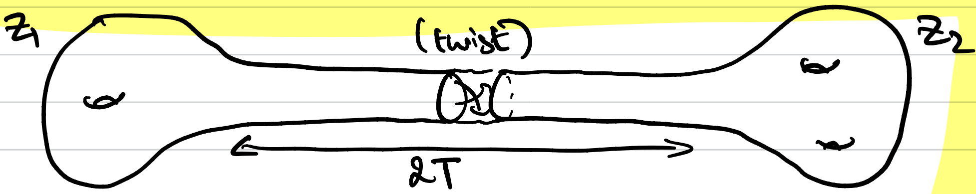
$|g-(d\rho^2+g_X)|=O(e^{-\mu\rho})$ $\quad$ $\mu>0$
$\oplus$ similar estimates for derivatives

EAC G_2 (X,ω_X,Ω_X) cpt CY_3. $|\varphi-(d\rho\wedge\omega_X+\mathrm{Re}\,\Omega_X)|=O(e^{-\mu\rho})$
$\oplus$ derivatives

eg. $CY_3\times S^1(/\Gamma)$, $X=T^2\times K3\,(/\Gamma)$ [CHNP '15]

Construction Start with a matching pair (Z_1,φ_1), (Z_2,φ_2)

gluing $T \gg 1$



(M_T,φ_T), $d\varphi_T = 0$, $\|d^*\varphi_T\| = O(e^{-\delta T})$

perturbation $T \gg 1 \rightarrow \exists!\ \tilde{\varphi}_T$ torsion-free, $[\tilde{\varphi}_T] = [\varphi_T]$, $\|\tilde{\varphi}_T - \varphi_T\|_{C^0} = O(e^{-\delta T})$

[Joyce]

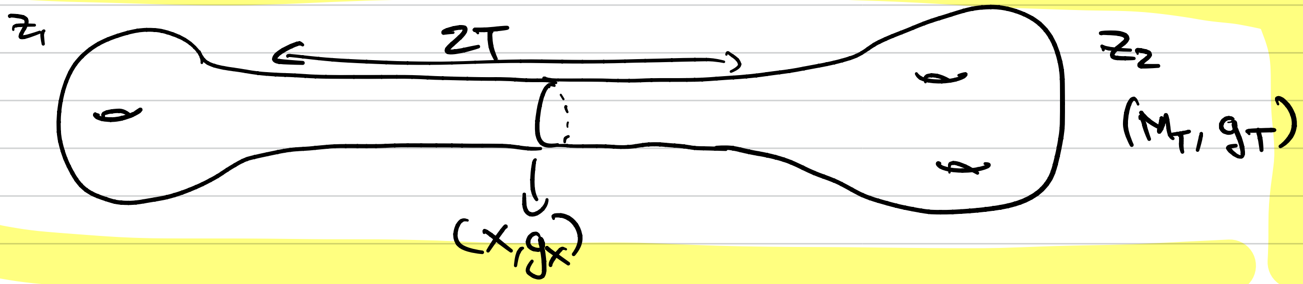
Fact $C \log(T - T_0) \leq d(\tilde{\varphi}_T, \tilde{\varphi}_{T_0}) \leq C'(T - T_0)$

Q towers of light states $\rightarrow$ decay of low $\lambda_n(T)$?
$\rightarrow$ density?
$\rightarrow$ interpretation/origin?

Q control of derivatives of $\tilde{\varphi}_T - \varphi_T$? Need improved estimates.

Prop $\forall k \in \mathbb{N}$, $\|\tilde{\varphi}_T - \varphi_T\|_{C^k} = O(e^{-\delta T})$ as $T \to \infty$.

Consequences
- can work with φ_T
- G_2 plays no role $\rightarrow$ tcs of Riemannian manifolds



Decay of low eigenvalues $\lambda_n^{(q)}(T)$ n-th eigenvalue of $\Delta \curvearrowright \Omega^q(M_T)$

Prop (L'23)
- $b^{q-1}(X) + b^q(X) = 0 \Rightarrow \lambda_1^{(q)}(T) \geq C$ for some cst
- $b^{q-1}(X) + b^q(X) \neq 0 \Rightarrow \exists\, C_1, C_n$ st:

$$\frac{C_1}{T^2} \leq \lambda_1^{(q)}(T) \leq \cdots \leq \lambda_n^{(q)}(T) \leq \frac{C_n}{T^2}$$

Density of low eigenvalues $s > 0$.

$$\Lambda_{q,\inf}(s) = \liminf_{T\to\infty} \#\{\text{eigenvalues of } \Delta_T \curvearrowright \Omega^q(M_T) \text{ in } (0, \tfrac{\pi^2 s}{T^2}]\}$$

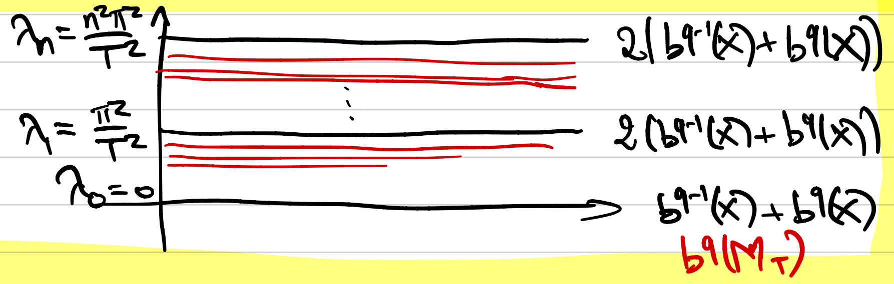
$$\Lambda_{q,\sup}(s) = \limsup_{T\to\infty} \# \text{ ———— }, \quad \Lambda^*_{q,\inf}(s),\ \Lambda^*_{q,\sup}(s).$$

Thm (L'23)

- If $b^{q-1}(X)+b^q(X)>0$ then
  $$\Lambda_{q,\sup}(s) = \Lambda_{q,\inf}(s)+O(1) = 2\left(b^{q-1}(X)+b^q(X)\right)\sqrt{s}+O(1).$$
- If $b^q(X)>0$ then
  $$\Lambda^*_{q,\sup} = \Lambda^*_{q,\inf}(s)+O(1) = 2\,b^q(X)\sqrt{s}+O(1).$$

Intrepretation

Density of low eigenvalues of Laplacian on $S^1_{2T}\times X$



tcs G_2 $\quad X = T^2\times K3$

physics limit $T\to\infty$ of (M_T,φ_T) $\rightsquigarrow$ physics on $S^1_{2T}\times T\times K3$ (duality with F-th...)

4) Idea of proof

Want
- $\to$ control on $|\tilde\varphi_T-\varphi_T|$ ①
- $\to$ eigenvalue estimates on Δ_T. ②

① Find $\tilde\varphi_T = \varphi_T + d\eta_T$, $d^*\eta_T = 0$ such that $d^{*_{\tilde\varphi_T}}\tilde\varphi_T = 0$.

$\hookrightarrow$ Non-linear PDE: use Banach fixed-pt thm (in appropriate functional spaces).

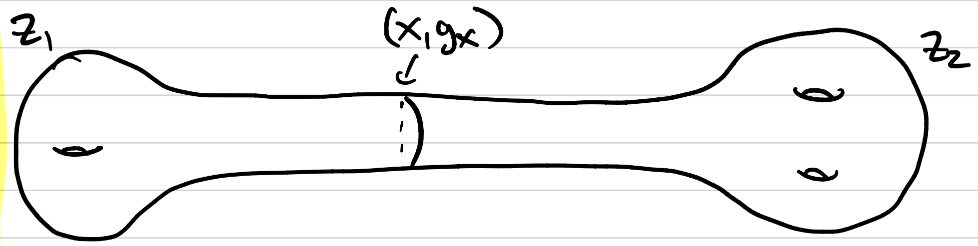
Need
- $d^*\varphi_T$ small ✓
- quadratic estimates on the non-linear part ✓
- "good" control on the linearisation $\longrightarrow$ here Δ_T

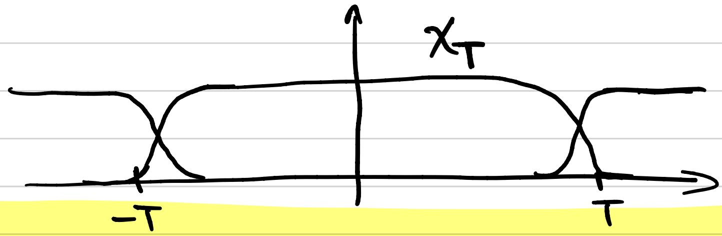
Elliptic regularity $\quad \eta\in\Omega^q(M)\cap\mathcal{H}^q(M)^\perp \Rightarrow \exists!\, u\in\Omega^q(M)\cap\mathcal{H}^q(M)^\perp$,
$\Delta u = \eta$ $\quad\oplus$ estimate
$$\|u\|_{L^p_{k+2}} \leq C(T)\,\|\Delta_T u\|_{L^p_k}$$

Q $\quad C(T)$? $\sim$ control the low eigenvalues.

Idea
- localise problems
- construct / approximate Δ_T^{-1}

Ansatz [Kapouleas '90] $\eta \in \Omega^q(M_T)$, seek u st $\Delta_T u = \eta$.





① $\eta_0 = \chi_T \eta \in \Omega^q(\mathbb{R}\times X)$ (supported in $[-T,T]\times X$).

↳ solve $(-\partial_t^2 + \Delta_X) u_0 = \eta_0$ first

② $\eta - \Delta_T(\chi_T u_0) = \eta_1 + \eta_2$, $\eta_i \in \Omega^q(Z_i)$ decays exponentially.

↳ solve $\Delta u_i = \eta_i$, $i = 1,2$. where u_i also decays exp.

If can do ① + ② → taking cutoffs, can solve $\Delta u_i = \eta + \tilde{\eta}$

$\|\tilde{\eta}\| \le C e^{-\delta T} \|\eta\|$.

Good case $b^{q-1}(X) + b^q(X) = 0$ then $-\partial_t^2 + \Delta_X : L^p_{k+2}(\Omega^p) \to L^p_k(\Omega^p)$ is invertible → ① has unique solution with estimates

[Kovalev-Singer '00] ② can be solved if $\eta \perp \mathcal{H}^q(M_T)$.

Bad case $b^{q-1}(X) + b^q(X) = 0$

① can be solved via separation of variables but solution is non-unique: $\Delta(u_0 + \xi) = \eta_0$

$\xi = t\alpha + \beta$, α, β translation-invt harmonic forms.

② Weighted spaces $L^p_{k,\delta} = L^p_k(e^{\delta\rho} d\mu)$

[Lockhart-McOwen '85]

Thm $0 < |\delta| \ll 1$ Then $\Delta : L^p_{k+2,\delta} \to L^p_{k,\delta}$ is Fredholm, and its image is the orthogonal space of $\ker \Delta \cap L^p_{-\delta}$.

pb too many obstructions

↳ find best ξ so $= \xi(\eta, T)$ so as to annihilate as many obstructions as possible.