Cohomogeneity one examples of deformed instantons

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Unicamp

Workshop BRIDGES: Special geometries and gauge theories

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Plan of talk

Words in the title:

- deformed instantons: special connections on complex line bundle $L \to M$: where M is either Kähler, G_2 or $Spin(7)$ manifold
- Nomenclature: Deformed HYM connections, deformed $G_2/Spin(7)$ instantons or deformed Donaldon-Thomas connections
- Cohomogeneity one: M/G is 1-dimensional

Plan:

- Background on G_2 geometry
- What are deformed instantons? Where they come from?
- On cohomogeneity one gauge theory
- Examples of dG_2-instantons
- Examples of d$Spin(7)$-instantons and dHYM connections

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G_2-structures on 7-manifolds

- A G_2-structure on M^7 is the data of a 3-form φ such that at each $p \in M$ $\exists \, \{x_i\}_{i=1}^7$ such that

$$\varphi\Big|_p = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}$$
$$= dx_1 \wedge (dx_{23} + dx_{45} + dx_{67}) + \mathrm{Re}((dx_2 + idx_3) \wedge (dx_4 + idx_5) \wedge (dx_6 + idx_7))$$

- $G_2 \subset SO(7) \Rightarrow \varphi$ determines a metric and orientation. Explicitly by

$$\frac{1}{6}(X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi = g(X, Y)\,\mathrm{vol}$$

- (M^7, φ) "is" G_2 manifold $\Leftrightarrow \nabla\varphi = 0 \Leftrightarrow d\varphi = 0 = d * \varphi$ (49 PDEs!)

- Trivial example: If (P^6, h, ω, Ω) is CY then $M^7 = S^1_t \times P^6$ is G_2 with $\varphi = dt \wedge \omega + \mathrm{Re}(\Omega)$ Hence $g = dt^2 + h$ and $*\varphi = \frac{1}{2}\omega^2 - dt \wedge \mathrm{Im}(\Omega)$.

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Why care about G_2-manifolds?

1. Admits a parallel spinor $\Rightarrow \mathrm{Ric}(g) = 0$: M^7 is an Einstein manifold
2. Calibrated geometry

Definition

A k-form η is a calibration on (M, g) if $d\eta = 0$ and $\eta(e_1, ..e_k) \leq 1$ $\forall e_i$ st $\|e_i\|_g = 1$.

If $L^k \subset M$ st $\eta\big|_L = \mathrm{vol}_L$ then L is calibrated submanifold. Suppose L is cpt calibrated and $\partial N^{k+1} = L \cup \overline{L'}$ then

$$Vol(L) = \int_L \eta = \int_{L'} \eta \leq Vol(L').$$

L is minimal k-submanifold in homology class! And L is minimiser! On (M^7, φ) G_2-manifold: calibrated by $\varphi(*\varphi)$ means $L^{3(4)}$ is (co)-associative. On (M^{2n}, ω, Ω) CY n-fold: calibrated by $\omega^k/k!$ means L^{2k} complex sub-mfd, calibrated by $\mathrm{Re}(e^{i\theta}\Omega)$ means L is special Lagrangian

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Gauge theory: G_2-instanton

3. On (M^4, g, vol), let A be a connection on a vector bundle $E \to M$ then $F_A := dA + \frac{1}{2}[A \wedge A] \in \Omega^2(End(E)) \cong \Omega^2_+ \oplus \Omega^2_-$. Call F_A an ASD instanton if $F_A \in \Omega^2_-$ i.e. $*F_A = -F_A$.

On (M^7, φ) we have $\Omega^2 \cong \Omega^2_7 \oplus \Omega^2_{14}$ since $\Omega^2 \cong \mathfrak{so}(7) \cong \mathbb{R}^7 \oplus \mathfrak{g}_2$. Call F_A a G_2-instanton if $F_A \in \Omega^2_{14}$ i.e. $*(F_A \wedge \varphi) = -F_A \Leftrightarrow F_A \wedge *\varphi = 0$.

Rmk: For a CY 3-fold P^6, instanton means (traceless) HYM i.e. $*(F_A \wedge \omega) = -F_A \Leftrightarrow F_A \wedge \mathrm{Im}(\Omega) = 0 = F_A \wedge \omega \wedge \omega \Leftrightarrow F_A \in \mathfrak{su}(3) = \Omega^{1,1}_0$

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Origin of deformed connections: Physics, SYZ conjecture,...

Mirror symmetry: $(P,\omega,\Omega) \longleftrightarrow (\check{P},\breve{\omega},\check{\Omega})$: Differential geometric version: SYZ conjecture: In certain limits (large J/vol), get dual special Lagrangian torus fibration st flat $U(1)$ connection $\mathbb{T}^3 \longleftrightarrow$ Point on $\check{\mathbb{T}}^3$.

MMMS, LYZ: Graph of $s : B^3 \to \check{P}$ is special Lagrangian iff Connection on P is dHYM i.e. $F_A \wedge \mathrm{Im}(\Omega) = 0$ and $F_A \wedge \frac{1}{2}\omega \wedge \omega = \frac{1}{6}F_A^3$ (phase 1). Generally: $\mathrm{Im}((\omega + iF_A)^3) = \tan(\theta)\mathrm{Re}((\omega + iF_A)^3)$ since $\mathrm{Re}(e^{i\theta}\Omega)$ is a calibration

GYZ, LL: Replace "SLag $\mathbb{T}^3$" by "co-associative $\mathbb{T}^4$". Then graph of $s : B^3 \to \check{M}$ is associative (+flat connection on B^3) iff Connection on M is deformed G_2: $F_A \wedge *\varphi = \frac{1}{6}F_A^3$.

Note: In the non-deformed case "F_A^3" term is zero!

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Mirror symmetry: $(P, \omega, \Omega) \longleftrightarrow (\check{P}, \check{\omega}, \check{\Omega})$: Differential geometric version: SYZ conjecture: In certain limits (large J/vol), get dual special Lagrangian torus fibration st flat $U(1)$ connection $\mathbb{T}^3 \longleftrightarrow$ Point on $\check{\mathbb{T}}^3$.

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Trivial examples of $\mathrm{d}G_2$ instantons

Trivial here means arising by pullback from lower dimensions!

Prop. On $M^7 = S^1 \times P^6$ as before, A is dHYM (phase 1) on $P^6 \Leftrightarrow A$ is $\mathrm{d}G_2$-instanton on M^7. Likewise: If M^4 is HK 4-manifold and take Riemannian product with $\mathbb{T}^3$. Then A is ASD-instanton iff $\mathrm{d}G_2$ (and also G_2) on $M^4 \times \mathbb{T}^3$.

By contrast to $\mathrm{d}G_2$, lots of dHYM examples are known, we have uniqueness and also existence results in many cases! Existence of solutions links to a notion of stability of L [Chen 20] (this verifies a mirror symmetry conjecture of Thomas-Yau). Crucial difference between $\mathrm{d}G_2$ and dHYM is that there is dd^c-lemma on Kähler manifolds! dHYM equation can be expressed as a Monge-Ampère type equation $F_A = F_{A_0} + dd^c f$!
No such thing in G_2-geometry :(

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What do we know about deformed G_2-instantons?

1. The dG_2-equation arises also the critical point of a Chern-Simons type functional: Let $\mathbf{A} := A_0 + t(A - A_0)$ on $M^7 \times [0,1]_t$ and consider
$$\mathcal{F}(\mathbf{A}) := \frac{1}{2}\int_{M\times[0,1]} \frac{1}{12}F_{\mathbf{A}}^4 - F_{\mathbf{A}}^2 \wedge *\varphi.$$
A is critical point of $\mathcal{F}$ iff it is dG_2. [Karigiannis-Leung 07]
2. The moduli space of dG_2-instantons on (M^7, φ) compact has expected dimension 0 i.e. they are discrete. [Kawai-Yamamoto 20]
3. The only known non-trivial examples are on nearly parallel G_2-manifolds, e.g. $SU(3)/U(1)$ and S^7, i.e. $\nabla\varphi \neq 0$ but instead $d\varphi = 4 * \varphi$ [Lotay-Oliveira 20]. They can be used to distinguish between isometric G_2-structures!
4. $\exists$ volume $V(A) = \int \sqrt{\det(Id + F_A^\sharp)}\,\mathrm{vol} \Rightarrow$ Gradient flow: Line bundle MCF [Jacob-Yau 14, Kawai-Yamamoto 21]. Critical points *contain* dG_2 and dG_2 are global minimisers. (only need cpt + $d\varphi = 0$!)

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Examples of G_2 manifolds

The possibility of G_2 metrics was first suggested by work of Berger 1950s

1. First local (non-trivial) examples: cones over NK [Bryant 87]
2. Complete examples: $\mathbb{R}^4 \times S^3$, $\Lambda^2_-(S^4)$, $\Lambda^2_-(\mathbb{CP}^2)$ [Bryant-Salamon 89]
3. Compact examples [Joyce, Kovalev, CHNP, J-K,..]
4. More non-compact examples [BGGG, BB, FHN,..]

Too hard to find examples on compact manifold. Try non-compact with maximal possible symmetry i.e. cohomogeneity one. Why? PDEs become ODEs!

Coho 1 implies that M/G is 1 dimensional i.e. $[0,1], S^1, [0,\infty)$, $(0,1)$
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Invariant connections on coho 1 manifolds

$M \simeq G/H_2 \cup G/H_1 \times \mathbb{R}^+$, where $H_1 \subset H_2 \subset G$ and H_2/H_1 is a sphere.

1. $\$(S^3) \cong \mathbb{R}^4 \times S^3 \simeq \frac{SU(2)^2}{SU(2)} \cup \frac{SU(2)^2}{1} \times \mathbb{R}^+$
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Find connections on principal orbit G/H_1. Given gp homomorphism $\lambda : H_1 \to K$ we define $K \hookrightarrow G \times_\lambda K \to G/H_1$. Canonical connection given by $d\lambda : \mathfrak{h}_1 \to \mathfrak{k}$ and all other connections are given by $\Lambda : (\mathfrak{m}, \mathrm{Ad}) \to (\mathfrak{k}, \mathrm{Ad} \circ \lambda)$ H-morphism and $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{m}$. When $K = U(1)$ i.e. $\mathfrak{k} \cong \mathbb{R}$, then

1. $\Lambda : 6\mathbb{R} \to \mathbb{R}$
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Conclusion: Only in the first case we can get families of $U(1)$ connections on principal orbit! Look for dG_2-instantons on $\mathbb{R}^4 \times S^3$ with Bryant-Salamon and BGGG metric.

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Extending solutions to singular orbit

Extending solutions from $G/H_1 \times \mathbb{R}^+$ to G/H_2. Need bundle extension, in the case of S^3 all bundles are trivial! [Eschenburg-Wang 00] gives condition for smooth extension at singular orbit. Get singular initial value problem!

Concretely:

A G-invariant tensor $T \in C^\infty(\otimes^p TM \otimes^q T^*M)$ can be identified with a 1-parameter family of H_2-equivariant map

$$T_t : V \supset S^k \to \otimes^p (V \oplus \mathfrak{p}) \otimes^q (V \oplus \mathfrak{p})$$

where $\mathfrak{g} = \mathfrak{h}_2 \oplus \mathfrak{p}$ and $S^k = H_2/H_1$ (working in normal bundle V of singular orbit!)

Each T_t is a finite sum of H_1-invariant tensors in $\otimes^p (V \oplus \mathfrak{p}) \otimes^q (V \oplus \mathfrak{p})$. For smoothness the coefficient functions must be even or odd depending on degree of the invariant tensor and lowest order term must be at least equal to degree of tensor.

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Abelian G_2-instantons

Smoothness condition for Bryant-Salamon and BGGG was worked out by Lotay-Oliveira. Using this they show

Lotay-Oliveira 16

For Bryant-Salamon G_2-instantons are $A = \frac{r^3 - c}{r} \sum_{i=1}^{3} c_i \eta_i$, and for BGGG they are $A = \frac{r^2 - 81/16}{r^2 - 9/16} c_1 \eta_1 + \frac{(r - 9/4)e^r}{\sqrt{r}(r + 9/4)^2}(c_2 \eta_2 + c_3 \eta_3)$, where r is distance to zero section, η_i are vertical 1-forms and $c_i \in \mathbb{R}$.

Rmk: Abelian G_2-instanton is a linear PDE, so ODE problem is much easier. Lotay-Oliveira also construct gauge $SU(2)$ G_2-instantons (which is non-linear)! Note that dG_2 is always non-linear although gauge is $U(1)$: $F_A \wedge *\varphi = \frac{1}{6} F_A^3$.

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For BGGG, a smooth dG_2-instanton is given by $A = f(r)\eta_1$, where $24\tan(f(r)/3 + c)f(r) = 16r^2 - 81$: $f \sim O(1)$. Here $c \in (0, \pi/2)$

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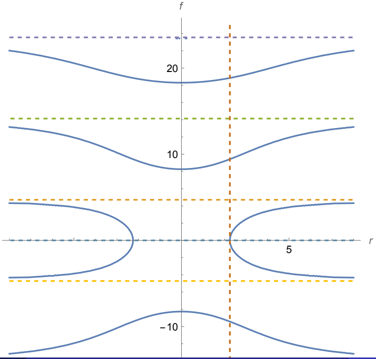

Relation betweeen G_2 and $\mathrm{d}G_2$

Recall $\mathrm{d}G_2$ means $F_A \wedge *\varphi = \frac{1}{6}F_A^3$. Note that to lowest order F_A is a G_2-instanton: put differently in the "large volume limit" $\mathrm{d}G_2$ is G_2!

We can make this precise:

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Given $\{A_k\}$ $\mathrm{d}G_2$ such that $c_k := \|F_{A_k}\|_\infty \to 0$ then define $B_k = A_k/c_k$ so that $\|F_{B_k}\| = 1$, the limit $B_k \to B_\infty$ is a smooth G_2-instanton.

Corollary

As $c \to \frac{\pi}{2}$, in BGGG case A_c convergence to flat connection. Rescaling and taking limit yields the example of Lotay-Oliveira.

One can construct local examples by evolving half-flat $SU(3)$-structures on nilmanifolds. [Chiossi-Salamon 02]

e.g on the Iwasawa manifold: We find examples of $\mathrm{d}G_2$ and G_2 instantons with very different asymptotic behaviour! Note: examples are incomplete!

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Let (M^8, Φ) be a $Spin(7)$ manifold, where Φ is the defining 4-form satisfying

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Example on a cone over S^7

As before, let's try to look for cohomogeneity one examples. Consider the spinor bundle of S^4: $\$(S^4) = \frac{Sp(2)}{Sp(1)Sp(1)} \cup \frac{Sp(2)}{Sp(1)} \times \mathbb{R}^+$ with Bryant-Salamon $Spin(7)$ metric.

Prop

The connection $A = f(r) \sum_{i=1}^3 c_i \eta_i$

$$f(r) = \frac{3r^2}{10} W(cr^{128/27})^{-1/2} : f \sim r^{2-\varepsilon},$$

where η_i are vertical 1-forms on S^3, gives an explicit solution on the Bryant-Salamon cone. W is the Lambert W function

Note: this is the cone over squashed S^7 not round one ($\mathrm{d}Spin(7)$ is $Spin(7)$ in the latter case with $Sp(2)Sp(1)$-symmetry and solution is $f = cr^2$)! This applies to any squashed 3-Sasakian e.g. $SU(3)/U(1)$!

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d$Spin(7)$ vs dHYM

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In dimension 8, we have $Sp(2) \subset SU(4) \subset Spin(7)$ i.e. hyperKähler $\subset$ Calabi-Yau $\subset$ $Spin(7)$.

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Thm [KY]

Suppose A is holomorphic wrt (ω, Ω). Then dHYM wrt ω (phase 1) iff d$Spin(7)$ wrt Φ.

Note: dHYM means $\mathrm{Im}((\omega + iF_A)^4) = \tan(\theta)\mathrm{Re}((\omega + iF_A)^4)$. Calabi showed that $T^*\mathbb{CP}^2$ admits a complete HK metric [Calabi ansatz]. This example is in fact cohomogeneity 1: $T^*\mathbb{CP}^2 = \frac{SU(3)}{U(1)SU(2)} \cup \frac{SU(3)}{U(1)} \times \mathbb{R}^+$

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Work in progress...

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where $r \in [\sqrt{2c}, \infty)$. One can define three different $\Phi_i = -\omega_i^2 + \omega_j^2 + \omega_k^2$. We have a 3-parameter family of connections

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Results in progress...

Thm...

A is dHYM wrt ω_1 iff $a_3 = a_4 = 0$ and $a_2(r)$ is defined by

$$\tan(\theta) = \frac{2(a_2 r^2 - 2ck)(a_2^2 - r^4/4 + c^2 - k^2)}{(a_2^2 - r^4/4 + c^2 - k^2)^2 - (a_2 r^2 - 2ck)^2}.$$

Rmk: $a_2 = 2ck/r^2$ is $Sp(2)$ connection [Hit] hence $Spin(7)$ instanton wrt Φ_2.

As comparison: You can play similar game on $F_{1,2} = SU(3)/\mathbb{T}^2$ with Kähler Einstein structure to get

$$\tan(\theta) = \frac{a_2(a_2^2 - k^2 - 3)}{3a_2^2 - k^2 - 1},$$

note that here a_2 is a fixed constant! Rmk: θ is fixed by a_2 and k because of compact setting!

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$$\tan(\theta) = \frac{2(a_2 r^2 - 2ck)(a_2^2 - r^4/4 + c^2 - k^2)}{(a_2^2 - r^4/4 + c^2 - k^2)^2 - (a_2 r^2 - 2ck)^2}.$$

Rmk: $a_2 = 2ck/r^2$ is $Sp(2)$ connection [Hit] hence $Spin(7)$ instanton wrt Φ_2.

As comparison: You can play similar game on $F_{1,2} = SU(3)/\mathbb{T}^2$ with Kähler Einstein structure to get

$$\tan(\theta) = \frac{a_2(a_2^2 - k^2 - 3)}{3a_2^2 - k^2 - 1},$$

note that here a_2 is a fixed constant! Rmk: θ is fixed by a_2 and k because of compact setting!

Results in progress...

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Results in progress...

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A is dHYM wrt $\Phi_1 = \frac{1}{2}(-\omega_1^2 + \omega_2^2 + \omega_3^2)$ iff either of the following holds:

$$A = 3k\theta_1 + \frac{6ck}{r^2}\theta_2,$$

or,

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Rmk: Some of these examples are in fact dHYM for instance wrt ω_2 (with phase 1: $C_3 = 0$). But more surprisingly this includes dHYM for all θ wrt ω_1. I have not yet simplified the solutions wrt Φ_2 :/

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Merci beaucoup pour votre attention!