Instantons on adjoint varieties

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$\boxed{\text{Intro}}$: Anti-Self-Dual bundles

$\left[Sp(n)\cdot Sp(1) = \frac{Sp(n)\times Sp(1)}{\mathbb{Z}_2} \right]$

§ Setting:

(M,g): quaternionic-Kähler mfld dim $4n$

$\uparrow$ $Hol(M) \subseteq Sp(n)\cdot Sp(1)$

Levi-Civita connection

$\left[\mathbb{R}^{4n} \cong \mathbb{H}^n \quad U(n,\mathbb{H}) = Sp_{2n}(\mathbb{C})\cap U(n) \quad Sp(n) = \{\phi \in GL_n(\mathbb{H}) \mid \phi^*(h) = h\} \quad \text{max cpct subgp} \subset Sp_{2n}(\mathbb{C}) \right]$

later on: compact, w/ positive (constant) scalar curvature

$\left[\nabla : \Gamma(TM)\otimes\Gamma(E) \to \Gamma(E) \text{ linear + Leibnitz} \right]$

ex $\mathbb{HP}^n$

$\rightsquigarrow$ $\exists$ $\mathcal{G} \subseteq End(TM)$ rank 3 subbundle ∇-invariant locally spanned by I, J, K almost cx structures $(\circ^2 = -id)$, g hermitian wrt I,J,K, $IJ = K$

$\left[\text{if scalar curvature} = 0 \Rightarrow HK \Rightarrow \nabla I = \nabla J = \nabla K = 0 \ (\Longleftrightarrow \text{Kähler \& complex mflds}) \right]$

Twistor space of (M,g): $Z : \{aI + bJ + cK \in \mathcal{G},\ a^2+b^2+c^2=1\} \xrightarrow{\pi} M$, fibre $S^2 \cong \mathbb{CP}^1$

ex $\mathbb{CP}^{2n+1} \longrightarrow \mathbb{HP}^n$

$[\{x_j, y_j\}_j] \longmapsto [\{x_j + i y_j\}_j]$

$\{$ positive quaternion-Kähler mflds $\}$ / isometries $\longleftrightarrow$ $\{$ complex contact Fano mflds w/ Kähler-Einstein metric $\}$ / biholo

[LeBrun, Salamon] $[Ric_g = \lambda g]$

contact Fano mfld: Fano: $-K_X$ ample

contact: $\exists$ $0 \to F \to TX \xrightarrow{\theta} L \to 0$

& $\theta_F : \Lambda^2 F \longrightarrow L$ is nowhere degenerate

Up to now: (M,g), Z.

§ a.s.d. bundles & "instantons"

$\mathbb{H} = \mathbb{C} + J\mathbb{C}$ | real str on $E \otimes \mathbb{H}$ induced by $(J\cdot)\otimes(J\cdot)$

$J\cdot : \mathbb{H}^n \to \mathbb{H}^n$

$$T^*M = (\underset{Sp(n)\circlearrowright}{E} \otimes \underset{Sp(1)\circlearrowleft}{\mathbb{H}})_{\mathbb{R}}$$

$\exists$ forme $\omega \in \Lambda^2 E$ invariante (d'où le nom "Sp")

$$\Lambda^2 T^*M = S^2E \otimes \underbrace{\Lambda^2\mathbb{H}}_{\text{trivial}} \oplus \Lambda^2 E \otimes S^2\mathbb{H} \cong$$

$$\cong S^2E \oplus S^2\mathbb{H} \oplus V$$

Rq | $\dim_{\mathbb{R}} 4 : V = 0$ intuition symétrique

[Capria-Salamon, Nitta] **def** | $w \in \Omega^2 T^*M$ self-dual (anti-self-dual) if $w \in \Gamma(S^2\mathbb{H})$ ($w \in \Gamma(S^2E)$)

$\nabla \in \Gamma(\mathrm{End}(E) \otimes T^*M)$ self-dual (anti-self-dual) if R^∇ is $\mathrm{End}(E)$-valued (a)sd (R^∇ = curvature 2-form $\in \Omega^2(\mathrm{End}(E))$)

recover self-dual in $\dim_{\mathbb{R}} = 4$: $*R^\nabla = \pm R^\nabla$

E v.b., ∇ connection

M compact $\rightsquigarrow$ $YM(\nabla) = \frac{1}{2}\int_M \|R^\nabla\|^2 \, d\mu$

Thm [Capria-Salamon, Nitta] | $YM(\nabla)$ minimized if R^∇ self-dual or anti-self-dual

E w/ ∇ connection (anti)self dual $\rightsquigarrow$ instantons

Ward correspondance [A-H-S]

$\left\{\begin{array}{l}\text{anti-self-dual bundles} \\ \text{on qK mflds } M \\ [+\text{ unitary structure } U(n)]\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\text{holo v.b. } \tilde{E} \text{ on } Z \text{ s.t.} \\ \bullet\ \tilde{E}|_{\pi^{-1}(x)} = \mathcal{O}_{\mathbb{P}^1} \quad \forall x \in M \\ \bullet\ [\exists \text{ iso } \tau : E \to \sigma^* E^* \text{ with } (\sigma^*\tau)^* = \tau \\ \Rightarrow \text{ positive hermitian form on sections of } E|_{\pi^{-1}(x)}]\end{array}\right\}$ real struct

" ASD bundles $\quad E \longmapsto \tilde{E} := \pi^* E$

(M, g), E asd $\longleftrightarrow$ $Z \downarrow M$ contact Fano, $\tilde{E}$ "instanton" $= \pi^* E$

qK pos compact

[structures unique up to bundle iso]

[$SU(n)$-structure $\longleftrightarrow$ $\det E$ = trivial]

[$Sp(n)$-structure $\longleftrightarrow$ $\mathrm{rk}\, E$ even, $\alpha : E \xrightarrow{\sim} E^*$, $\alpha^* = -\alpha$, $(\sigma^*\alpha)^* \circ \tau\alpha = \tau$

[$SO(n)$-structure $\longleftrightarrow$ " , $\alpha^* = \alpha$ + compatibility w/ τ]

$\mathbb{C}P^3 \xrightarrow{\pi} \mathbb{H}P^1$

$\tilde{E} = \pi^* E \longrightarrow E$ and

[Drinfeld-Manin, Rawnsley, Douady, Hitchin] : $H^1(\mathbb{P}^3, \tilde{E}(-2)) = 0$

Moreover, one supposes: $c_1(\tilde{E}) = 0$ (SU(k)-structure on E)

Thm [A-D-H-M] | $\tilde{E}$ is the cohomology of :

$\mathbb{M}_{\mathbb{P}^3}$: $0 \to W \otimes \mathcal{O}(-1) \xrightarrow{a} V \otimes \mathcal{O} \xrightarrow{b} W^\vee \otimes \mathcal{O}(1) \to 0$

* uses that $H^0(\mathbb{P}^3, \tilde{E}(-1)) = 0$ $= H^2(\mathbb{P}^3, \tilde{E}(-3))$ + more vanishings in cohomology +

~~...~~ Beilinson spectral sequence

$\Downarrow$

$\left[E^{p,q} \Longrightarrow \begin{cases} E & \text{degree } 0 \\ 0 & \text{otherwise} \end{cases} \right.$

$\left. E_1^{p,q} = H^q(E \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) \otimes \mathcal{O}_{\mathbb{P}^n}(p) \right]$

Rq • Beilinson s.s. $\Longleftrightarrow$ $\exists$ <u>full semiorthogonal decomposition</u>

der. cat. of bounded complexes of coherent sheaves $\to$ $D^b(\mathbb{P}^n) = \langle \mathcal{O}(-1), \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n-1) \rangle$

FSOD: $E_1, \dots, E_n \in D^b(X)$

$\mathrm{Ext}^\bullet(E_i, E_i) = \mathbb{C}\,\mathrm{id}[0]$

Beilinson Type s.s. $\Leftarrow$ $\mathrm{Ext}^\bullet(E_j, E_i) = 0 \quad \forall j > i$

$+ \langle E_1, \dots, E_n \rangle = D^b(X)$

Rq •

Moreover, one can study the moduli space of instantons $\mathcal{M}_{\mathbb{P}^3}$

e.g. W dim 1, V dim 4 (charge 1 instantons)

b surj $\Rightarrow \exists$ exact sequence $0 \to W(-1) \xrightarrow{a'} Q^\vee \to \tilde{E} \to 0$

$a' \in \mathrm{Hom}(W(-1), Q^\vee) \cong \Lambda^2 \mathbb{C}^4$

$\overline{\mathcal{M}_{1,\mathbb{P}^3}} \subseteq \mathbb{P}(\Lambda^2 \mathbb{C}^4)$ w/ degenerations along a quadric

Nagatomo's homogeneous monads

complex Fano contact manifolds $\longleftrightarrow$ projectivised cotangent bundle

adjoint varieties

Let G complex Lie gp, $\mathfrak{g}$ Lie algebra

$G \curvearrowright \mathfrak{g}$ def adjoint var : minimal G-orbit $Z \subseteq \mathbb{P}(\mathfrak{g})$

[all known w/ $\mathrm{Pic}(Z) = \mathbb{Z}L$]

Examples

C_{n+1} $G = Sp_{2n+2}(\mathbb{C}) \rightsquigarrow Z = v_2(\mathbb{P}^{2n+1}) \subseteq \mathbb{P}\, S^2(\mathbb{C}^{2n+2})$ $= \mathbb{P}\,\mathfrak{g}$

$\downarrow$

$M = \mathbb{H}\mathbb{P}^n$

A_n $G = SL_{n+1}(\mathbb{C}) \rightsquigarrow Z = \mathbb{P}\,T^*\mathbb{C}\mathbb{P}^n$

$\downarrow$

$M = G(2, n+1)$

$B_\bullet, D_\bullet$ $G = SO_{n+4}(\mathbb{C}) \rightsquigarrow Z = G(\mathbb{C}\mathbb{P}^1, Q^{n+2})$

$\downarrow$

$M = \widetilde{G}(\mathbb{R}^4, \mathbb{R}^{4+n})$

G_2 $G = G_2 \rightsquigarrow Z = G_2/P_1$ "{null planes inside $\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$}"

$\downarrow$

$M = G_2^c/SO(4)$ "{$\mathbb{H} \subseteq \mathbb{O}$}"

max. cpct subgp

$(G_2/P_1 \subseteq G(2, \mathbb{C}^7)$ $\leftarrow$ zero locus of a section of a v.b.$)$

F_4, E_n "Wolf spaces"

Rq adjoint var are homogeneous $G \curvearrowright Z$ transitive

on homogeneous space, $\exists$ homogeneous v.b. : $\widetilde{E}$ v.b. on Z w/ $G \curvearrowright \widetilde{E}$ compatible w/ $G \curvearrowright Z$

classified combinatorially in terms of representations

$Z = G/P$ {homogeneous v.b.} $\xleftrightarrow{1:1}$ {P-representations} $\lambda \mapsto \widetilde{E}_\lambda$

subgp {irr. hom. v.b.} $\xleftrightarrow{1:1}$ {irreducible P-representations} $\xleftrightarrow{1:1}$ {P-weights}

Theorem [Nagatomo] Classification of irreducible homogeneous bundles on Z which are "instantons", i.e. the pullback of an anti-self-dual homogeneous bundle on M

↑ M also admits an action of $G_{\mathbb{R}} \subseteq G$ — maximal compact real subgp

$\tilde{E} = \tilde{E}_\lambda$, λ P-weight s.t. $K(\lambda, \Theta^\vee) = 0$ — highest coroot

$\Rightarrow G$-weight

Moreover, he finds a class of weights λ ($K(\lambda, \Theta^\vee) = 1$ & $\lambda \geq 0$) s.t.

$\exists$ monad $0 \to \tilde{E}_{\lambda^\vee} \to V_\lambda \to \tilde{E}_\lambda \to 0$

s.t. cohom of monad is an "instanton" homogeneous bundle

G representation of weight λ

Examples

$\mathbb{P}^3$ ← $G = Sp_4(\mathbb{C})$ ← $\exists\, w \in \Lambda^2 \mathbb{C}^4$, $Sp_4(\mathbb{C}) = Sp_4(w)$

$$0 \to \mathcal{O}(-1) \xrightarrow{a} \mathcal{O}^{\oplus 4} \xrightarrow{b} \mathcal{O}(1) \to 0$$

b surj $\Rightarrow$ $0 \to \mathcal{O}(-1) \xrightarrow{a} Q^\vee \to \tilde{E} \to 0$

$\tilde{E} = $ "$Q^\vee / \mathcal{O}(-1)$" is a Sp_4-homogeneous bundle of

$a = w|_{\mathcal{O}(-1)}$ & $\tilde{E}$ = null correlation bundle

$\rightsquigarrow \mathcal{M}_{\mathbb{P}^3} \subseteq \mathbb{P} H^0(\mathrm{Hom}(\mathcal{O}(-1), Q^\vee)) \cong \mathbb{P}\Lambda^2 \mathbb{C}^4$

$pt = \mathbb{P}(w) \subseteq \mathbb{P}(\Lambda^2 \mathbb{C}^4)$

$Sp_4(\mathbb{C}) \circlearrowright$ $\quad$ $Sp_4(\mathbb{C}) \circlearrowright$

$\mathbb{P}(T^* \mathbb{P}^n)$ $G = SL_{n+1}(\mathbb{C})$ $\rightsquigarrow$ different kinds of monads (depending on the chosen Chern class)

$\mathbb{P}(T^*\mathbb{P}^n) \subseteq \mathbb{P}^n \times \mathbb{P}^{n\vee}$

e.g. Nagatomo's monad: $0 \to \mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1) \to \mathcal{O}^{2n+2} \to \mathcal{O}(1,0) \oplus \mathcal{O}(0,1) \to 0$

$\rightarrow$ cohomology $\rightsquigarrow$ homogeneous v.b. on $\mathbb{P}T^*\mathbb{P}^n$

G_2 $\quad G_2/P_1 \subseteq G(2,\mathbb{C}^7)$

$\mathcal{U}$ Tautological rank 2 bundle on $G(2,\mathbb{C}^7)$ restricted to G_2/P_1

Nagatomo's monad : $\quad 0 \to \mathcal{U} \to \mathcal{O}^{\oplus 7} \to \mathcal{U}^* \to 0$

cohomology = homogeneous instanton $= S^2 U^*(-1)$.

2 problems :
i) • completeness mod. sp. of instantons
(every instanton w/ fixed invariants (eg Chern classes) comes from one of Nagatomo's monad construction ?)

ii) • study of moduli space via monads (eg $M_{\mathbb{P}^3} \subseteq \mathbb{P}\Lambda^2\mathbb{C}^4$) & and their compactifications (GIT quotients)

① case G_2 (Nagatomo revisited)

Thm (Nagatomo) | completeness for 1-instantons on G_2/P_1, i.e. pull back of anti-self-dual bundles C^∞-isomorphic to $S^2U^*(-1)$.
& every 1-instanton is the cohomology of a monad $0 \to U \to \mathcal{O}^{\oplus 7} \to U^* \to 0$

Proof revisited
① $\quad \mathbb{P}T^*\mathbb{P}^2 \subseteq G_2/P_1 = Z \quad$; restriction of instantons are instantons + classif. of instantons on $\mathbb{P}T^*\mathbb{P}^2$ [Buchdahl]

$\quad\quad \downarrow \quad\quad\quad\quad \downarrow$

covering family of subvarieties $\quad \mathbb{P}^2 \subseteq G_2^{\mathbb{C}}/SO(4) = M$

① cohomology vanishings (generalize Hitchin ∀ twistor space)

Thm [Na, NN] $\quad Z \to M^{\dim 4n}$ $\quad E$ asd, $\tilde{E} = \pi^* E$, contact str. on Z given by $L = \mathcal{O}(2)$

i) $H^i(Z, \tilde{E}(k)) = 0$ if $i+k+1<0$ $\quad 1 \le i \le n$

ii) $H^1(Z, \tilde{E}(-2)) = 0$, $H^2(Z, \tilde{E}(-3)) = 0$ if $n \ge 2$

iii) $H^{2n-1}(Z, \tilde{E}(-2n+1)) = 0$ if $n \ge 2$, $H^{2n}(Z, \tilde{E}(-2n)) = 0$

iv) $H^i(Z, \tilde{E}(k)) = 0$ if $i+k>0$ $\quad 2n \ge i \ge n+1$

① + ② for G_2/P_1

$$\Rightarrow H^*(\underset{\mathrm{Hom}(\mathcal{U}^*(1),\tilde{E})}{\tilde{E}\otimes\mathcal{U}(-1)}) = H^*(\underset{\mathrm{Hom}(\mathcal{O}(1),\tilde{E})}{\tilde{E}(-1)}) = H^*(\underset{\mathrm{Hom}(\mathcal{O}(2),\tilde{E})}{\tilde{E}(-2)}) = 0$$

Now, on G_2/P_1 $\exists$ full sed [Kuznetsov]

③ $D^b(G_2/P_1) = \langle \mathcal{U}, \mathcal{O}, \mathcal{U}^*, \mathcal{O}(1), \mathcal{U}^*(1), \mathcal{O}(2) \rangle$

vanishings $\Rightarrow$ $\tilde{E} \subseteq \langle \mathcal{U}, \mathcal{O}, \mathcal{U}^* \rangle$ (extension)

more effort: $H^i \tilde{E}\otimes\mathcal{U} = H^i \tilde{E}\,\mathcal{U}^* = 0$ if $i \neq 1$

$\Rightarrow \tilde{E} \underset{\text{q-iso}}{\simeq}$ is the cohom of $0 \to \mathcal{U} \to \mathcal{O}^{\oplus 7} \to \mathcal{U}^* \to 0$

$\blacksquare$

Project: completeness for monads on other adjoint varieties:

$\mathbb{P}^{2n+1}$ ✓, $\mathbb{P}T^*\mathbb{P}^n$ ✓, G_2/P_1 ✓

B_n D_n : ① $\dots \subseteq G(\mathbb{C}\mathbb{P}^1, Q^{n+3}) \subseteq G(\mathbb{C}\mathbb{P}^1, Q^{n+4}) \subseteq \dots$

② [Na, NN]

③ [Kuznetsov, Smirnov] full sed for $D^b Z$ for all adjoint varieties

[~~Kuznetsov, ...~~]

$D^b OG(2,2n+1) = \langle A, \dots, A(2n-3) \rangle$

$A = \langle \mathcal{O}, \mathcal{U}^*, S^2\mathcal{U}^*, \dots, S^{n-2}\mathcal{U}^*, S \rangle$

eg $D^b OG(2,2n) = \langle \mathcal{U}^{2W_{n-1}}(-1), \mathcal{U}^{2W_n}(-1), A, B(1), \dots, B(n-2), A(n-1), \dots, A(2n-4) \rangle$

$A := \langle \mathcal{O}, \mathcal{U}^\vee, S^2\mathcal{U}^\vee, \dots, S^{n-3}\mathcal{U}^\vee, S_-, S_+ \rangle$

$B := \langle \text{———}, S^{n-2}\mathcal{U}^\vee, S_-, S_+ \rangle$