PONCELET CURVES AND SURFACES

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18 /09/ 2012

CIRCLES AND TRIANGLE

- One triangle gives uniquely two circles, one inscribed C(r), the other circumscribed C(R).
- Then there are infinitely many such triangles inscribed in C(R) and circumscribed around C(r).
- Two random circles are not the circumscribed and the inscribed circles of a triangle.

PORISM

- This kind of result is called *porism:*

  it does not occur in general but if it occurs, it occurs for infinitely many cases.

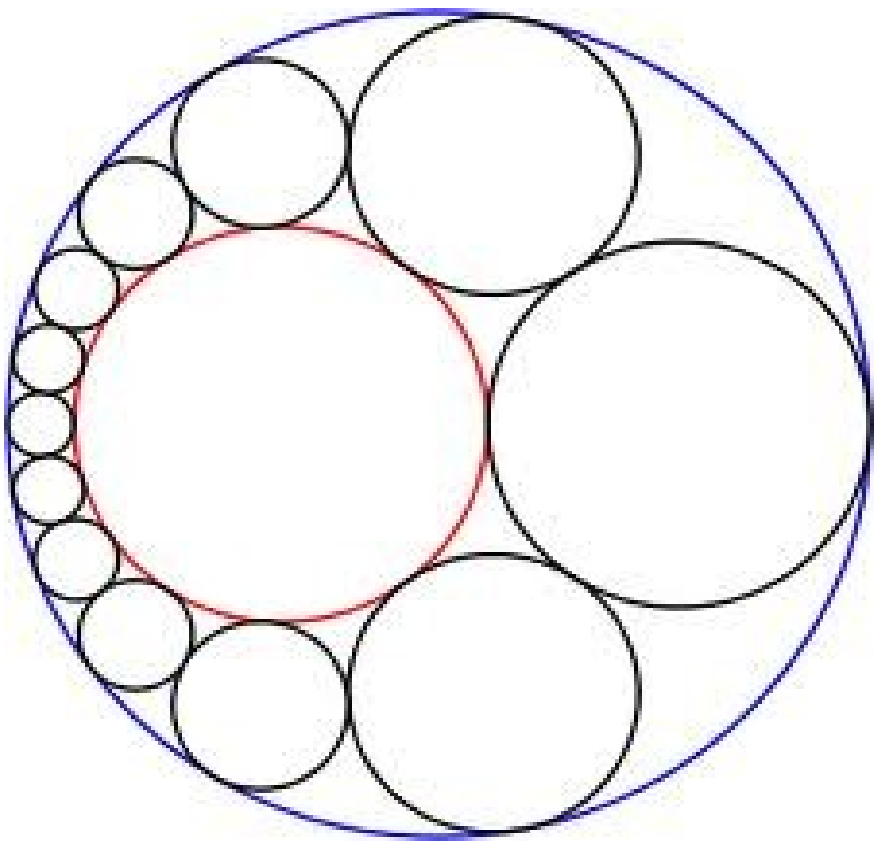
- The first known result of this type is the so-called Chapple (or Euler-Chapple 1746) formula for a triangle inscribed in C(R) and circumscribed around C(r): $2rR=R^2-d^2$.

- For a quadrilateral inscribed in C(R) and circumscribed around C(r) it is done by Steiner (1827) :

$$1/(R-d)^2+1/(R+d)^2=1/r^2.$$

STEINER PORISM

It can be solved by an inversion that sends the blue and red circles on two concentric ones.



PONCELET PORISM

About Chapple result Poncelet said:

« It's a projective result. »

It means that it should be also true for two general conics.

It leads to the Poncelet porism. It is more complicated than Steiner's porism.

NAPOLEON IN RUSSIA

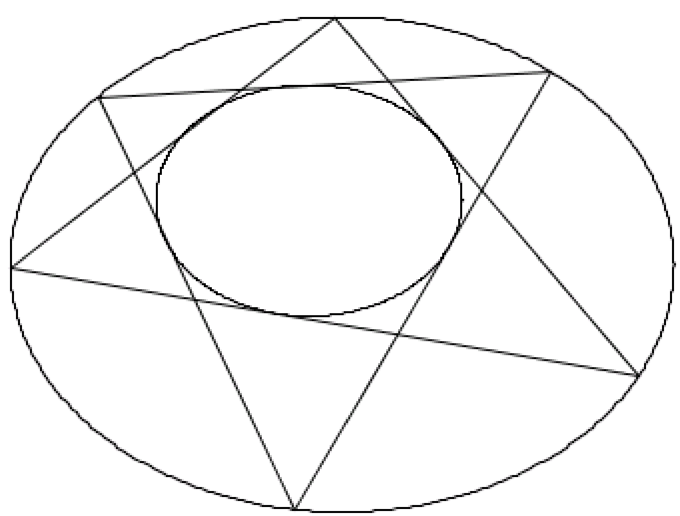
- Poncelet held prisoner in Saratov after the Krasnoi's battle.
- He gave the basis of modern algebraic geometry : duality, conservation of number, continuity principle.

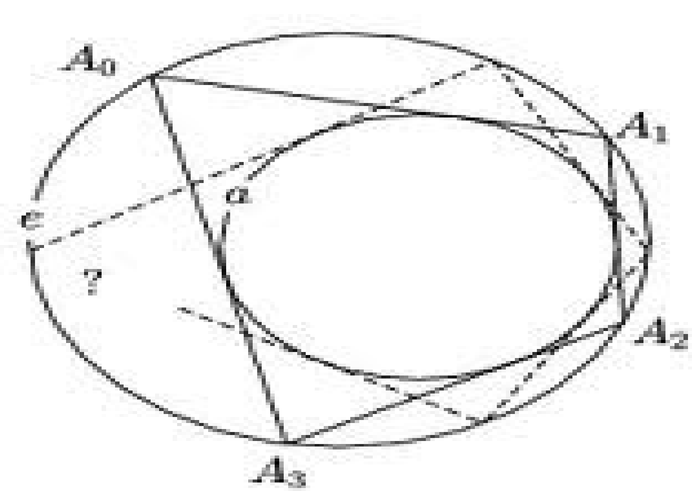
Il neigeait. On était vaincu par sa conquète.
Pour la première fois l'aigle baissait la tête.
Sombres jours ! l'empereur revenait lentement,
Laissant derrière lui bruler Moscou fumant.
Il neigeait. L'âpre hiver fondait en avalanche.
Après la plaine blanche une autre plaine blanche.
On ne connaissait plus les chefs ni le drapeau.
Hier la grande armée, et maintenant troupeau.
On ne distinguait plus les ailes ni le centre.
Il neigeait. Les blessés s'abritaient dans le ventre
Des chevaux morts ;

PONCELET'S CLOSURE THEOREM

In jail at Saratov, he wrote his famous closure theorem:

THM (1814). *Let C and D be two conics on the complex projective plane such that there exists one n-gon inscribed in D and circumscribed around C. Then there are infinitely many such n-gons.*





THREE PROOFS OF PONCELET'S THEOREM

- By Poncelet himself : n-gon tangent to n conics

  and specialize to two conics.
- By Jacobi : involutions on elliptic curve.
- By Weyr : pencil of quadrics in $\mathbf{P}^3$.

Our aim today is to propose another one based on vector bundle techniques.

SECANT VARIETY OF RATIONAL CURVE

- Rational normal curve, its secant variety, its associated bundle. A section of the last one corresponds to a hyperplane section of the first one : the $n(n-1)/2$ vertices of a complete tangent n-gon.
- A Poncelet curve : a pencil of sections

  (remove two lines in the ad-hoc matrix).
- It gives a proof of Darboux theorem.

PONCELET CURVE

- A Poncelet curve associated to a conic C is a degree *(n-1)* curve passing through the *n(n-1)/2* vertices of a complete *n*-gon.
- Theorem(Darboux). *Let S be a curve of degree (n-1). If there is a complete n-gon tangent to a smooth conic C and inscribed into S, then there are infinitely many of them*.

PROOF OF PONCELET PORISM

- An inscribed *n*-gon : *n* points on *C*. It remains

  $n(n-1)/2-n$ points. This remaining set belongs to

  E of degree ($n-3$), s.t. $S=C \cup E$ is a Poncelet curve.
- So any point of *C* is a vertex of a complete *n*-gon inscribed in *S*. By Bézout's theorem this *n*-gon has *n* vertices on *C* (if not the number of vertices on *E* is $> -n+n(n-1)/2$ which is impossible for degree reason).

WHICH CURVE IS A PONCELET CURVE

- All the conics.
- All the cubics.
- Only a divisor of quartics : Luroth quartics.

  This divisor has degree 54 (Morley 1919).
- It is the first Donaldson number q13 on $\mathbf{P}^2$.

  (Morley, Le Potier, Ottaviani-Sernesi)

PONCELET SURFACES

- Set of planes 3-secant to rational normal curve leads to Poncelet surfaces.
- Net of degree *n* divisors on $\mathbf{P}^1$ = surface of degree $n-2$

CUBIC PONCELET SURFACES

- <u>Theorem</u> : *A cubic surface S in* $\mathbf{P}^3$ *is a Poncelet surface.*
- Proof : 6 points associated to *S.* Triple tensor. They have a resolution by a persymmetric matrix.

  Smooth cubic and

  its 27 lines
- (Salmon-Cayley 1849)