Freeness of line arrangements

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Abstract

In this work in common with Daniele Faenzi we describe the logarithmic bundle associated to a arrangement of lines in $\mathbb{P}^2(\mathbb{C})$ as the image of the ideal sheaf of its dual set of points in $\mathbb{P}^{2\vee}(\mathbb{C})$ by the Fourier-Mukai transform. This bundle is called free when it splits as a sum of two line bundles. Terao conjectures (in 1981) that the freeness of this bundle depends only on the combinatorics of the arrangement of lines. Thanks to this new description, we recover all the known results and moreover we give a list of new cases such that the Terao's conjecture is true.

1 Freeness of line arrangements

Hyperplane arrangements in $\mathbb{P}^n(\mathbb{C})$ or simply line arrangements in $\mathbb{P}^2(\mathbb{C})$ is a deep subject; it concerns topology, geometry, combinatorics, complex analysis and so on. A good introduction to this subject is [2] and a complete and comprehensive text is the Orlik-Terao's book [10].

Today we will focus on line arrangements in the complex projective plane. A line arrangement in $\mathbb{P}^2 = \mathbb{P}(\mathbb{C}[x_0, x_1, x_2])$ is a finite collection of lines, say $\{l_1, \cdots, l_s\}$. The union of these lines is a reduced divisor denoted by $D = \{f = 0\}$, where f is the product of the s linear forms defining the l_i's. Saito (see [5]) associates to D the bundle $T(\log D)$ of vector fields with logarithmic poles along D (more or less the dual of the logarithmic bundle introduced by Deligne in 1970). It is a vector bundle of rank 2, defined by the following exact sequence:

$$0 \longrightarrow T(\log D) \longrightarrow \mathcal{O}_{\mathbb{P}^2}^3 \xrightarrow{(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})} \mathcal{O}_{\mathbb{P}^2}(s-1).$$

The sheaf image of the last map is the jacobian ideal $\mathcal{J}_{\mathrm{sing}(f)}(s-1)$ of f.

$$0 \longrightarrow T(\log D) \longrightarrow \mathcal{O}_{\mathbb{P}^2}^3 \xrightarrow{(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})} \mathcal{J}_{\mathrm{sing}(f)}(s-1) \longrightarrow 0.$$

Set theoretically $\mathrm{sing}(f)$ is the set of singular point of the divisor D. For instance if D consists in s distinct lines then $sing(f)$ is the set of $\binom{s}{2}$ vertices of D. Such a arrangement is called *generic*.

In 1980 Terao introduces the notion of freeness for a hyperplane arrangement (cf. [7]). The arrangement D is called *free* when $T(\log D)$ splits as a sum of two line bundles. More precisely

Définition 1.1. *D is free with exponents (a, b) with $0 \leq a \leq b$, when*

$$T(\log D) = \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b).$$

We recall that according to Grothendieck theorem $T(\log D)$ splits on any line $l \simeq \mathbb{P}^1$ i.e. that $T(\log D) \simeq \mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b)$ with $a + b = -c_1(T(\log D))$. But in general it does not split on the whole space. For instance, when $s \geq 4$ the generic arrangement is not free and is very far to be free. In some sense it could be the opposite notion. It is not easy to produce free arrangements. We have to impose a lot of triple points comparatively to the number of lines. Let me give four examples.

Exemple 1.2. *When D consists in four line in general position the bundle $T(\log D)$ is $\Omega_{\mathbb{P}^2}$ which of course is not a sum of line bundles. The set* $\mathrm{sing}(f)$ *is the set of six vertices. When D is defined by the divisor $xyz(x + y) = 0$ the scheme* $\mathrm{sing}(f)$ *is supported by three double points and a triple point.*

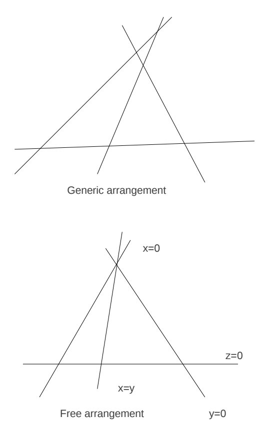

Figure 1: Four lines with six dictint double points or with a triple point.

We have the following global relation:

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}-3z\frac{\partial f}{\partial y}=0.$$

This global relation gives an injective map $\mathcal{O}_{\mathbb{P}^2}(-1)\hookrightarrow T(\log D)$ *and by a simple computation of Chern classes we find*

$$T(\log D)\simeq \mathcal{O}_{\mathbb{P}^2}(-1)\oplus \mathcal{O}_{\mathbb{P}^2}(-2).$$

In general we will not be able to explicit a global relation. We have to characterize freeness with other methods.

Exemple 1.3. *With six lines the scheme* $\mathrm{sing}(f)$ *has length* 15. *When* D *is defined by*

$$xyz(x-y)(x-z)(y-z)=0$$

there are 4 *triple points (so length* 12*) and* 3 *double points. Then*

$$T(\log D)\simeq \mathcal{O}_{\mathbb{P}^2}(-2)\oplus \mathcal{O}_{\mathbb{P}^2}(-3).$$

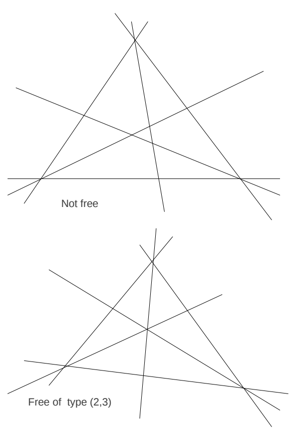

Figure 2: Six lines free and not free.

When we have 3 *instead of* 4 *triple points it is not free.*

Exemple 1.4. *But the most famous example of free arrangement is the one corresponding to the Hesse configuration of* 12 *lines through the nine inflexion points of a smooth plane cubic. These nine points are defined by* *the intersection of a smooth cubic $f = x^3 + y^3 + z^3 = 0$ and its hessian curve $H(f) = xyz = 0$. In the pencil $(f, H(f))$ there are four singular cubics, more precisely four triangles $x^3 + y^3 + z^3 - 3axyz = 0$ with $a = \infty, 1, j, j^2$. Then the arrangement of 12 lines has 9 quadruple points and 12 double points in correspondance with any side of the triangles. Indeed in any triangle the choice of one side gives an opposite vertex. It is free with exponents $(4, 7)$. Moreover the dual set of 9 lines in $\mathbb{P}^{2\vee}$ with 12 triple points is also free with exponents $(4, 4)$.*

This set of 9 points is the well known obstruction to extend the Sylevster-Gallai theorem from $\mathbb{R}$ to $\mathbb{C}$. This problem was proposed by Sylvester in 1893, then by Erdos in 1943 and solved on $\mathbb{R}$ by Gallai in 1944 and with a simplest proof by Kelly in 1948. In 1966 Serre wonder if any SGC over $\mathbb{C}$ is planar. Once again it is solved by Kelly in 1986. It implies that for a finite set of points non aligned on the complex projective plane there exits at least one two-secant or one three-secant.

These examples show that freenees is related to the existence and the number of triple points. But it is possible to have the same number of lines in D and $D^{'}$ and the same number of triple points (count with multiplicity) with D free and $D^{'}$ not.

Exemple 1.5. *The divisor D consists in 10 lines, four through a point A, five through a point B and the line (AB). The scheme of triple points has degree $16 = \binom{4}{2} + \binom{5}{2}$ and $T(\log D)$ is free with exponents $(4, 5)$. Assume now that D consists in 10 lines, three of them pass through a point and the remaining seven lines pass through another point. Then the scheme of triple points is also of degree 16 but $T(\log D)$ is not free. I will explain after why it is not free.*

So if we want to produce free arrangements we need to characterize more precisely its combinatorics. In order to do it one associates to an arrangement an INTERSECTION LATTICE.
We will say that two intersection lattices $G = (V, E)$ and $G^{'} = (V^{'}, E^{'}))$ are isomorphic if there is a bijection between vertices E and $E^{'}$ and edges V and $V^{'}$ that conserve the incidence between edges and vertices.

2 Terao's conjecture

The main open question about these bundles (also valid on $\mathbb{P}^n$, for $n \geq 2$) is Terao's conjecture (see [10]):

Conjecture 1 (Terao). *Freeness depends only on combinatorics*

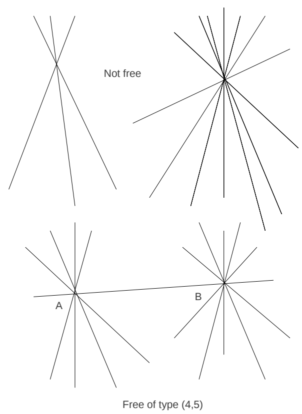

Figure 3: Ten lines free and not free.

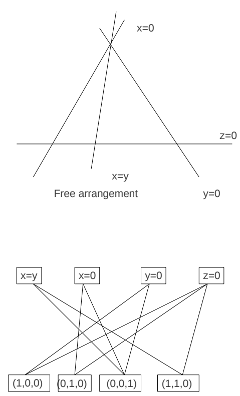

Figure 4: Intersection lattice.

By combinatorics we mean that two arrangements with the same (modulo isomorphism) intersection lattice are both free or both non free. It can be formulated also in the following way

Conjecture 2 (Terao)**.** *D and $D^{'}$ are two arrangements with the same combinatorics and D is free. Then $D^{'}$ is free with the same exponents, i.e. $T(\log D) = T(\log D^{'})$.*

Remarque 2.1. *The conjecture does not extend to non free arrangements. More precisely we do not have D and $D^{'}$ have the same combinatorics $\Leftrightarrow$ $T(\log D) = T(\log D^{'})$.*

Indeed, six lines tangent to a smooth conic leads to a vector bundle SL(2) invariant such that every line l tangent to the smooth conic decomposes the bundle in the form $\mathcal{O}_l(-1) \oplus \mathcal{O}_l(-4)$. When the six lines $\{l_1, \cdots, l_6\}$ are not tangent to a smooth conic the bundle $T(\log D)$ decomposes in the form $\mathcal{O}_l(-2) \oplus \mathcal{O}_l(-3)$ for all $l \neq l_i, i = 1, \cdots, 6$. In other terms both bundles are different even if the arrangements have the same combinatorics.

The conjecture is proved for a number of line ≤ 10. We propose a new appr oach to attack this conjecture. Our ideas are certainly inspired by the work of Schenck. Thanks to this approach we obtain all the known results easily and a little more.

3 Our approach

3.1 Fourier Mukai transform

We propose a new approach based on projective duality and vector bundles technics. Any line of the divisor D corresponds to a point in $\mathbb{P}^{2\vee}$. This way we associate to D a finite set of points Z in $\mathbb{P}^{2\vee}$. From now, in order to insist on the correspondence, we will denote by $Z \subset \mathbb{P}^{2\vee}$ the finite set of points and by $D_Z \subset \mathbb{P}^2$ the corresponding divisor.
Let us now introduce the variety $\mathbb{F} \subset \mathbb{P}^2 \times \mathbb{P}^{2\vee}$. which is the incidence variety *point-line* in $\mathbb{P}^2$, and the projections p and q on $\mathbb{P}^2$ and $\mathbb{P}^{2\vee}$.

$$\begin{array}{ccc} \mathbb{F} & \xrightarrow{q} & \mathbb{P}^{2\vee} \\ {\scriptstyle p}\downarrow & & \\ \mathbb{P}^2 & & \end{array}$$

Let $\mathcal{J}_Z$ be the ideal sheaf of Z in $\mathbb{P}^{2\vee}$. We are interested by the triple points of D_Z. By duality it is the scheme of three secant lines to Z. This scheme is defined by the sheaf $R_Z := R^1 p_* q^* \mathcal{J}_Z(1)$. First of all we prove that the Saito vector bundle $T(\log D_Z)$ is obtained by looking at $\mathcal{J}_Z(1)$ on $\mathbb{P}^2$. More precisely, we prove:

Theorem 3.1. *$p_* q^* \mathcal{J}_Z(1) \simeq T(\log D_Z)$.*

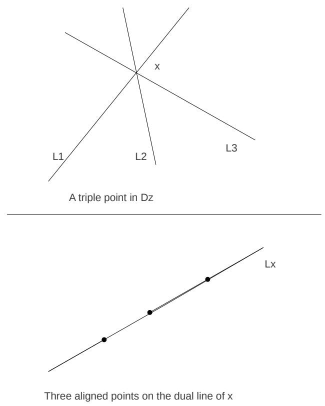

Figure 5: Projective duality.

Taking the resolution of $\mathbb{F}$ in the product $\mathbb{P}^2 \times \mathbb{P}^{2\vee}$ we obtain:

$$0 \longrightarrow T(\log D_Z) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{|Z|-1} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}^{|Z|-3} \longrightarrow R_Z \longrightarrow 0.$$

The matrix M is a matrix of linear forms, R_Z is the scheme defining the triple points of D_Z. According to this exact sequence we have:

$$c_1(T(\log D_Z) = 1- \mid Z \mid, c_2(T(\log D_Z) = \binom{\mid Z \mid -1}{2} - \mid R_Z \mid .$$

It appears that two bundles $T(\log D_Z)$ and $T(\log D_{Z'})$ have the same Chern classes if and only both arrangements have the same number of lines and of triple points (with multiplicity)..

If D_Z is free with exponents (a, b) we have $c_1(T(\log D_Z)) = -(a+b), c_2(T(\log D_Z) = ab$. So to be free with exponents (a, b) it is necessary to have $\binom{a}{2} + \binom{b}{2}$ triple points.

Three Advantages: 1) The canonical sequences involving $\mathcal{J}_Z$ give results for $T(\log D_Z)$. For instance the exact sequence, when $l^\vee \in Z$

$$0 \longrightarrow \mathcal{J}_Z(1) \longrightarrow \mathcal{J}_{Z\setminus\{l^\vee\}}(1) \longrightarrow \mathcal{O}_{l^\vee} \longrightarrow 0.$$

leads to the so-called addition-deletion theorem of Ziegler (i.e. link between $D, D \setminus \{l\}$ and $D_{|l}$.

2) It gives also a simple way to construct free arrangements (of course not all the free arrangements are of this kind).

Theorem 3.2 (inductively free)**.** *Let D_Z a set of N points. Let $t \geq 0$ and let L be a line passing through $t+1$ points of Z exactly. Both propositions are equivalent:*

- *Z is free with exponents $(t, N-t-1)$.*
- *Any k-secant line to $Z_1 = Z \setminus L \cap Z$, for $k \geq 2$, is $(k+1)$-secant to Z.*

Remarque 3.3. *The conjecture is true for these arrangements. They are determined by their combinatorics.*

Proof. The reduction of Z along L gives

$$0 \longrightarrow \mathcal{J}_{Z_1} \longrightarrow \mathcal{J}_Z(1) \longrightarrow \mathcal{O}_L(-t) \longrightarrow 0.$$

Applying the functor p_*q^* to this exact sequence gives:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(t+1-N) \longrightarrow p_*q^*\mathcal{J}_Z(1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-t)$$

$$\longrightarrow R^1p_*q^*\mathcal{J}_{Z_1} \longrightarrow R^1p_*q^*\mathcal{J}_Z(1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}/\mathcal{J}_{L^\vee}^{t-1} \longrightarrow 0.$$

When any k-secant to Z_1 is $k+1$-secant to Z then the long exact sequence splits in two short exact sequences and we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(t+1-N) \longrightarrow p_*q^*\mathcal{J}_Z(1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-t) \longrightarrow 0.$$

It proves that Z is free with exponents $(t, N-t-1)$. $\square$

Exemple 3.4. *In this exemple we see that D_Z consisting in 9 lines is free with exponents $(3,5)$.*

3) But the main point is the link between curves through Z and splitting of $T(\log D_Z)$ on lines in $\mathbb{P}^2$.

Theorem 3.5. *The following conditions are equivalent:*
i) $T_Z \otimes \mathcal{O}_{x^\vee} = \mathcal{O}_{x^\vee}(-n) \oplus \mathcal{O}_{x^\vee}(-n-r)$.
ii) $H^0((\mathcal{J}_Z \otimes \mathcal{J}_x^n)(n+1)) \neq 0$ and $H^0((\mathcal{J}_Z \otimes \mathcal{J}_x^{n-1})(n)) = 0$.

Remenber the example number 4. The set Z consists in 3 points on a line and 7 points on another line such that the intersection point does not belong to Z. The Chern classes of $T(\log D_Z)$ are the Chern classes of $\mathcal{O}_{\mathbb{P}^2}(-4) \oplus \mathcal{O}_{\mathbb{P}^2}(-5)$ but from any general point x there exists a curve of degree four and multiplicity 3 at x passing through Z. Then the decomposition on $x^\vee$ is

$$\mathcal{O}_{x^\vee}(-3) \oplus \mathcal{O}_{x^\vee}(-6).$$

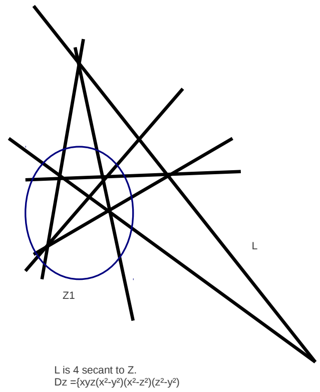

Figure 6: Projective duality.

It does not coincide with the decomposition of the free arrangement, then it is not free.

On the contrary when Z consists in 10 points, 6 on one line and 5 on another line (this time the intersection point belongs to Z), the arrangement is free according to the supersolvability theorem (take the line 5 secant).

3.2 Splitting of bundles on lines

The main theorem is a general result concerning rank two vector bundles on the projective plane proved by Elencwajg. It generalizes Yoshinaga's result ([13], Thm. 2.2).

Theorem 3.6 (Elencwajg)**.** *Let E be a rank two vector bundle on $\mathbb{P}^2$. Assume that $c_t(E) = c_t(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-r))$ (where c_t is the total Chern class) and that there exists one line $l \subset \mathbb{P}^2$ verifying $E_l = \mathcal{O}_l \oplus \mathcal{O}_l(-r)$. Then $E = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-r)$.*

The following corollary (in fact a contraposition) is very useful.

Corollaire 3.7. *Let E be a rank two vector bundle on $\mathbb{P}^2$. The following conditions are equivalent:*
i) $c_t(E) = c_t(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-r))$ *and* $E \neq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-r)$.
ii) $\forall l \subset \mathbb{P}^2,\ \exists t > 0$ *such that* $E_l = \mathcal{O}_l(t) \oplus \mathcal{O}_l(-t-r)$.
iii) $\exists t_0 > 0$ *such that* $h^0(E(-t_0)) \neq 0$.

Let us call r_Z the number of distinct triple points in D_Z (or the number of distinct 3-secant to Z).

Corollaire 3.8. *Assume that $c_t(T_Z(n)) = c_t(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-r))$, then,*

$$r_Z > (n-1)^2 \Rightarrow D_Z \textit{ is free with exponents } (n, n+r).$$

Proof. We must prove that $H^0(T_Z(n-1)) = 0$. If not, there are three degree $(n-1)$ forms vanishing along r_Z points (since $M(p_i) = (0)$ for $i = 1, \cdots, r_Z$), where

$$0 \longrightarrow T_Z \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}^3 \longrightarrow \mathcal{J}_{\mathrm{sing}(f)}(2n) \longrightarrow 0.$$

Of course we can assume that $H^0(T_Z(n-2)) = 0$ so that the three forms do not have a commun factor. Then they vanish along a scheme of length less or equal to $(n-1)^2$ by Bézout's theorem. $\square$

Exemple 3.9 (Dual Hesse)**.** *They are exactly 12 triple points in the divisor of degree nine D_Z. Computing the Chern classes of $T(\log D_Z)$ we find $c_t(T_Z(4)) = c_t(\mathcal{O}_{\mathbb{P}^2}^2)$. Since the number r_Z of triple points is strictly bigger than $(4-1)^2 = 9$ the corollary 3.8 proves that D_Z is free with exponents $(4, 4)$.*

The following corollary generalizes a result by Wakefield and Yuzvinsky. They proved that the Terao conjecture holds when there exists a a line $l \subset \mathbb{P}^{2\vee}$ such that $\mathrm{card}(l \cap Z) \geq \frac{\mathrm{card}(Z)-1}{2}$.

Corollaire 3.10. *Assume that $c_t(T(\log D_Z)(n)) = c_t(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-r))$ and that there is a point in D_Z with multiplicity $\geq n$ ($\geq n-1$ in $\mathbb{R}$). Then, D_Z is free with exponents $(n, n+r)$.*

Proof. If Z is not free with exponents $(n, n+r)$ then, by the corollary 3.7, $h^0(T_Z(n-1)) \neq 0$.
Let us consider the line L which is n secant to Z.
The reduction of Z along L gives

$$0 \longrightarrow \mathcal{J}_{Z_1} \longrightarrow \mathcal{J}_Z(1) \longrightarrow \mathcal{O}_L(1-n) \longrightarrow 0.$$

Applying the functor p_*q^* to this exact sequence gives:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-n-r-1) \longrightarrow T_Z \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1-n) \longrightarrow$$

$$\longrightarrow R^1p_*q^*\mathcal{J}_{Z_1} \longrightarrow R^1p_*q^*\mathcal{J}_Z(1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}/\mathcal{J}_{L^\vee}^{n-2} \longrightarrow 0.$$

We have a short exact sequence where Γ is the set of $k \geq 2$ secant to Z_1 that are exactly k-secant to Z:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-n-r-1) \longrightarrow T_Z \longrightarrow \mathcal{J}_\Gamma(1-n) \longrightarrow 0.$$

The length of Γ is $c_2(T_Z(n+r+1)) = 1+r$.

Let us twist the exact sequence above by $(n-1)$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-r-2) \longrightarrow T_Z(n-1) \longrightarrow \mathcal{J}_\Gamma \longrightarrow 0.$$

Our hypothesis $h^0(T_Z(n-1)) \neq 0$ implies $h^0(\mathcal{J}_\Gamma) \neq 0$ which is impossible. $\square$

Exemple 3.11 (Hesse Arrangement)**.** *The arrangement D of 12 lines passing through the nine points of inflexion of a smooth cubic has 9 quadruple points and 12 double points. It proves that $c_t(T(\log D)) = c_t(\mathcal{O}_{\mathbb{P}^2}(-4) \oplus \mathcal{O}_{\mathbb{P}^2}(-7))$ (66 double points = 9*6+12). Since it exists at least one quadruple points we can conclude with corollary 3.10 that it is free with exponents $(4, 7)$.*

In the real case[1], with the same kind of technics, we have proved the Terao's conjecture when there exits a point with multiplicity $\geq n-1$ for combinatorics of a free arrangement with exponents $\mathcal{O}_{\mathbb{P}^2}(-n) \oplus \mathcal{O}_{\mathbb{P}^2}(-n-r)$ (using a theorem in Aigner-Ziegler's book on the slopes). In this case, and for the first time, we use the existence of a free model. Until now it was not necessary, in particular for less than 10 lines.

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